

# INTRODUCTION TO HESSENBERG VARIETIES

MIKIYA MASUDA

The flag variety  $\text{Fl}(n)$  is a smooth projective variety consisting of nested sequences of complex linear subspaces of  $\mathbb{C}^n$ :

$$\text{Fl}(n) := \{V_\bullet = (V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}} V_i = i \ (\forall i)\}.$$

It has a natural cell decomposition by Schubert cells. The Schubert varieties are closures of Schubert cells and there is a rich study for them.

Hessenberg varieties are a new family of subvarieties of the flag variety finally defined by De Mari-Procesi-Shayman in 1992. This family contains previously known varieties such as Springer fibers related to geometric representation theory, Peterson varieties related to the quantum cohomology of  $\text{Fl}(n)$ , and permutohedral varieties which are smooth toric varieties. Given a square matrix  $A$  of order  $n$  and a function  $h$  from  $[n] = \{1, \dots, n\}$  to itself satisfying

$$h(1) \leq h(2) \leq \cdots \leq h(n) \quad \text{and} \quad h(j) \geq j \quad (\forall j \in [n]),$$

the Hessenberg variety  $\text{Hess}(A, h)$  associated to  $A$  and  $h$  is defined by

$$\text{Hess}(A, h) := \{V_\bullet \in \text{Fl}(n) \mid AV_i \subset V_{h(i)} \quad (\forall i \in [n])\}$$

where the matrix  $A$  is regarded as a linear operator from  $\mathbb{C}^n$  to itself. The definition of  $\text{Hess}(A, h)$  is simple but the topology and geometry is complicated. It has been intensively studied in these ten years but there are many things to be studied.

In this five talks, I will discuss its cohomology and relations to hyperplane arrangements, GKM theory, and Stanley-Stembridge conjecture on graph theory. The title of each talk is as follows:

- (1) Flag varieties and Hessenberg varieties
- (2) Cohomology of regular nilpotent Hessenberg varieties and hyperplane arrangements
- (3) Regular semisimple Hessenberg varieties and GKM theory
- (4) Cohomology of regular semisimple Hessenberg varieties as  $\mathfrak{S}_n$ -modules and chromatic symmetric functions on graphs
- (5) Automorphism groups of regular semisimple Hessenberg varieties

OSAKA CENTRAL ADVANCED MATHEMATICAL INSTITUTE, OSAKA METROPOLITAN UNIVERSITY  
Email address: mikiyamsd@gmail.com