Stochastic PDEs involving a bilaplacian operator

Suprio Bhar,
Department of Mathematics and Statistics,
Indian Institute of Technology Kanpur, India.

Department of Mathematics,
IIT Madras

June 4, 2024

This talk is based on a joint work with Barun Sarkar (arXiv:2312.16550v1)

Motivation

Bilaplacian operator appears in many physical models:

- Membrane models[1] [2]
- Sandpiles model[3]

[1] Noemi Kurt, *Maximum and entropic repulsion for a Gaussian membrane model in the critical dimension*, Ann. Probab., 37(2):687–725, 2009.

[2] Alessandra Cipriani, Biltu Dan, and Rajat Subhra Hazra, *The scaling limit of the membrane model* Ann. Probab., 47(6):3963–4001, 2019.

[3] Alessandra Cipriani, Rajat Subhra Hazra, and Wioletta M. Ruszel, *Scaling limit of the odometer in divisible sandpiles*. Probab. Theory Related Fields, 172(3-4):829–868, 2018.

Problem statement

Let $\mathcal{S}$ be the space of rapidly decreasing smooth functions on $\mathbb{R}$, with the dual space $\mathcal{S}'$, the space of tempered distributions.

Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions and a 2-dimensional standard Brownian motion $\{B_t\}_{t\geq 0}$, we are interested in the existence and uniqueness of strong solutions of the Stochastic PDE (SPDE) in $\mathcal{S}'$:

$$dX_t = L(X_t)\, dt + A(X_t) \cdot dB_t, t \geq 0; \quad X_0 = \Psi, \tag{1}$$

and the associated PDE in $\mathcal{S}'$:

$$\frac{\partial}{\partial t} u_t = L(u_t), t \geq 0; \quad u_0 = \Psi, \tag{2}$$

where $\Psi \in \mathcal{S}'$ and

Operators L and A

$L : \mathcal{S}' \to \mathcal{S}', A : \mathcal{S}' \to \mathcal{S}' \times \mathcal{S}'$ are linear differential operators defined as follows:

$$L(\phi) := -\frac{\kappa^2}{2}\,\partial^4\,\phi + \frac{\sigma^2}{2}\,\partial^2\,\phi - b\,\partial\,\phi, \tag{3}$$

and $A(\phi) := (A_1(\phi), A_2(\phi))$, with $A_1, A_2 : \mathcal{S}' \to \mathcal{S}'$, such that

$$A_1(\phi) := -\sigma\partial\phi, \quad A_2(\phi) := \kappa\partial^2\phi, \tag{4}$$

with κ, σ, b being real constants. We write $AX_t \cdot dB_t = A_1X_t\,dB_t^1 + A_2X_t\,dB_t^2$.

Outline

1. Tempered Distributions and Hermite-Sobolev Spaces
2. A Monotonicity inequality for the pair (L, A)
3. Existence and uniqueness of Strong solutions to the Stochastic PDE
4. Applications to PDEs

Outline

Tempered Distributions

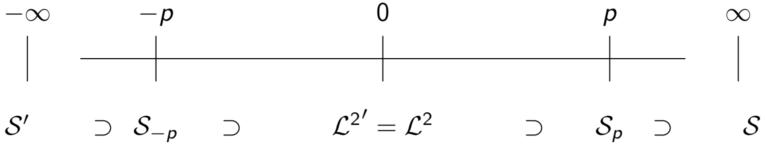
- Let $\mathcal{S}$ denote the space of real valued rapidly decreasing smooth functions on $\mathbb{R}$ (Schwartz class) with dual $\mathcal{S}'$, the space of tempered distributions.
- For $p \in \mathbb{R}$, consider the increasing norms $\|\cdot\|_p$, defined by the inner products

$$\langle f, g \rangle_p := \sum_{k=0}^{\infty} (2k+1)^{2p} \langle f, h_k \rangle \langle g, h_k \rangle, \quad f, g \in \mathcal{S}.$$

Here, $\{h_k\}_{k=0}^{\infty}$ is an orthonormal basis for $\mathcal{L}^2(\mathbb{R}, dx)$ given by Hermite functions $h_k(t) := (2^k k! \sqrt{\pi})^{-1/2} \exp\{-t^2/2\} H_k(t), t \in \mathbb{R}$, where H_k are the Hermite polynomials.

Hermite-Sobolev spaces

The Hermite-Sobolev spaces[4] $\mathcal{S}_p, p \in \mathbb{R}$ are defined to be the completion of $\mathcal{S}$ in $\|\cdot\|_p$. It can be shown that $(\mathcal{S}_{-p}, \|\cdot\|_{-p})$ is isometrically isomorphic to the dual of $(\mathcal{S}_p, \|\cdot\|_p)$ for $p \geq 0$.



[4] Kiyosi Itô, *Foundations of stochastic differential equations in infinite-dimensional spaces*, volume 47 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1984.

A recursion for the derivative of Hermite functions

$$\partial\, h_n(x) = \sqrt{\frac{n}{2}}\, h_{n-1}(x) - \sqrt{\frac{n+1}{2}}\, h_{n+1}(x),\ \forall x \in \mathbb{R}, n = 0, 1, \cdots .$$

Derivative operator

- Given a tempered distribution $\psi \in \mathcal{S}'$, the distributional derivative of ψ is defined via the following relation

$$\langle \partial\psi\,,\, \phi\rangle := -\,\langle \psi\,,\, \partial\phi\rangle\,,\ \forall \phi \in \mathcal{S}.$$

- $\partial : \mathcal{S}_p \to \mathcal{S}_{p-\frac{1}{2}}$ is a bounded linear operator. So the Laplacian $\triangle = \partial^2$ is a bounded linear operator from $\mathcal{S}_p$ to $\mathcal{S}_{p-1}$.
- $L : \mathcal{S}_p \to \mathcal{S}_{p-2}, A_1 : \mathcal{S}_p \to \mathcal{S}_{p-\frac{1}{2}}$ and $A_2 : \mathcal{S}_p \to \mathcal{S}_{p-1}$ are bounded linear operators.

Outline

Theorem (A Monotonicity inequality for the pair (L, A) (B., Sarkar))

Fix $p \in \mathbb{R}$. Then, there exists a constant $C = C(p, \kappa, \sigma, b) > 0$, such that

$$2 \langle \phi\, ,\, L\phi \rangle_p + \|A\phi\|^2_{HS(p)} \leq C\|\phi\|^2_p, \forall \phi \in \mathcal{S}, \tag{5}$$

where $\|A\phi\|^2_{HS(p)} := \|A_1\phi\|^2_p + \|A_2\phi\|^2_p, \forall \phi \in \mathcal{S}$. Moreover, by density arguments, the inequality is true for all $\phi \in \mathcal{S}_{p+2}$.

A quick overview of known results

Remark

This inequality for Stochastic PDEs in Hilbert spaces was first considered in (Krylov and Rozovskii 1979)[a].

For second order L and first order A with constant coefficients, the inequality in the setting of Hermite-Sobolev spaces was first proved in (Gawarecki, Mandrekar and Rajeev 2009)[b]

Some extensions of the above result, was proved in (Bhar and Rajeev 2015) and (Bhar, Bhaskaran and Sarkar 2020)

---

[a] N. V. Krylov and B. L. Rozovskiĭ, *Stochastic evolution equations*. In *Current problems in mathematics, Vol. 14 (Russian)*, pages 71–147, 256. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1979.

[b] L. Gawarecki, V. Mandrekar, and B. Rajeev, *The monotonicity inequality for linear stochastic partial differential equations*. Infin. Dimens. Anal. Quantum Probab. Relat. Top., 12(4):575–591, 2009.

Remark (Integration by Parts)

For $\phi \in \mathcal{S}$, $\left(-\left\langle \phi\,,\, \partial^4 \phi \right\rangle_0 + \|\partial^2\phi\|_0^2\right) = 0$ *etc..*

Idea of Proof

$$2\left\langle \phi\,,\, L\phi \right\rangle_p + \|A\phi\|_{HS(p)}^2 = \kappa^2 \left(-\left\langle \phi\,,\, \partial^4 \phi \right\rangle_p + \|\partial^2\phi\|_p^2\right) + \left(\left\langle \phi\,,\, \sigma^2\, \partial^2\, \phi - 2b\, \partial\, \phi \right\rangle_p + \| - \sigma\partial\phi\|_p^2\right).$$

Write $\phi = \sum_{n=0}^{\infty} \phi_n h_n$ and expand the terms using the recurrence relation for the derivatives of Hermite functions.

Outline

Definition

Let $p \in \mathbb{R}$ and Ψ be an $\mathcal{S}_p$-valued $\mathcal{F}_0$ measurable random variable. We say that an $(\mathcal{F}_t)_{t\geq 0}$ adapted $\mathcal{S}_p$ valued continuous process $\{X_t\}_t$ is a strong solution of the Stochastic PDE if it satisfies the equality

$$X_t = \Psi + \int_0^t L(X_s)\, ds + \int_0^t A(X_s) \cdot dB_s, t \geq 0 \tag{6}$$

in some $\mathcal{S}_q$ with $q \leq p$. In this case, we say that $\{X_t\}_t$ is an $\mathcal{S}_p$ valued strong solution with equality in $\mathcal{S}_q$.

Theorem (B., Sarkar)

Let $p \in \mathbb{R}$ and Ψ be an $\mathcal{S}_p$-valued $\mathcal{F}_0$ measurable random variable such that $\mathbb{E}\|\Psi\|_p^2 < \infty$. Then, there exists a unique $\mathcal{S}_p$ valued solution of the Stochastic PDE with equality in $\mathcal{S}_{p-2}$.

Idea of Proof

Application of Theorem 1 from (Gawarecki, Mandrekar and Rajeev 2008)[a]

[a] L. Gawarecki, V. Mandrekar, and B. Rajeev, *Linear stochastic differential equations in the dual of a multi-Hilbertian space*. Theory Stoch. Process., 14(2):28–34, 2008.

Outline

Connections with PDEs

Consider the stochastic PDE in $\mathcal{S}_p$:

$$X_t = \Psi + \int_0^t A(X_s).\, dB_s + \int_0^t L(X_s)\, ds, t \geq 0$$

with $\Psi \in \mathcal{S}_p$. Then,

$$\mathbb{E}X_t = \Psi + \int_0^t L(\mathbb{E}X_s)\, ds, t \geq 0$$

Remark

Our approach is motivated by (Rajeev and Thangavelu 2003)[a], where they show

$$\mathbb{E}(\delta_{B_t}) = \delta_0 + \int_0^t \frac{1}{2}\partial^2(\mathbb{E}\delta_{B_s})\, ds, t \geq 0.$$

This yields the fundamental solution to the heat equation.

[a] B. Rajeev and S. Thangavelu, *Probabilistic representations of solutions to the heat equation*. Proc. Indian Acad. Sci. Math. Sci., 113(3):321–332, 2003.

Definition

Let $p \in \mathbb{R}$ and $\Psi \in \mathcal{S}_p$. We say that an $\mathcal{S}_p$ valued continuous $\{u_t\}_t$ is a strong solution of the PDE if it satisfies the equality

$$u_t = \Psi + \int_0^t L(u_s)\, ds, t \geq 0 \tag{7}$$

in some $\mathcal{S}_q$ with $q \leq p$. In this case, we say that $\{u_t\}_t$ is an $\mathcal{S}_p$ valued strong solution of the PDE with equality in $\mathcal{S}_q$.

Theorem (B., Sarkar)

Let $p \in \mathbb{R}$ and $\Psi \in \mathcal{S}_p$. Then, there exists a unique $\mathcal{S}_p$ valued strong solution $\{u_t\}_t$ of the PDE with equality in $\mathcal{S}_{p-2}$. Moreover, $u_t = \mathbb{E}X_t, \forall t \geq 0$, where $\{X_t\}_t$ is the strong solution of the Stochastic PDE.

Remark

The uniqueness follows from the Monotonicity inequality for (L, A).

Thank You