Stochastic partial differential equations and invariant manifolds in embedded Hilbert spaces

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Invariant submanifolds

1. Invariant manifolds in finite dimensions
2. Stochastic partial differential equations and invariant manifolds in embedded Hilbert spaces
3. Semilinear stochastic partial differential equations
4. Interplay between SPDEs and finite dimensional SDEs
5. Finite dimensional diffusions

Invariant manifolds in finite dimensions

Stochastic differential equations

- Consider the $\mathbb{R}^d$-valued SDE

$$\left\{ \begin{array}{rcl} dX_t & = & b(X_t)dt + \sigma(X_t)dW_t \\ X_0 & = & x_0. \end{array} \right. \qquad (1)$$

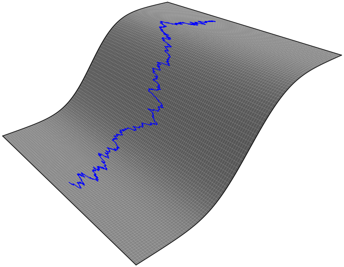
- Here $x_0 \in \mathbb{R}^d$ is the starting point.
- We consider measurable mappings

$$b : \mathbb{R}^d \to \mathbb{R}^d \quad \text{and} \quad \sigma : \mathbb{R}^d \to \mathbb{R}^{d \times r}.$$

- W is an $\mathbb{R}^r$-valued standard Wiener process.

Invariant manifolds

- Let $\mathcal{M}$ be an m-dimensional C^2-submanifold of $\mathbb{R}^d$ ($m \leq d$).
- $\mathcal{M}$ is called *locally invariant* for the SDE (1) if for each $x_0 \in \mathcal{M}$ there exists a local weak solution $(\mathbb{B}, W, X)$ with $X_0 = x_0$ such that $X^\tau \in \mathcal{M}$ for some stopping time $\tau > 0$.
- Trajectory on an invariant submanifold:



Classical invariance result

- Recall the $\mathbb{R}^d$-valued SDE

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t \\ X_0 = x_0. \end{cases} \qquad (2)$$

- We assume that $b \in C(\mathbb{R}^d; \mathbb{R}^d)$ and $\sigma \in C^1(\mathbb{R}^d; \mathbb{R}^{d \times r})$.

Theorem 1

$\mathcal{M}$ is locally invariant for the SDE (2) if and only if

$$b(x) - \frac{1}{2} \sum_{j=1}^{r} D\sigma^j(x)\sigma^j(x) \in T_x\mathcal{M},$$

$$\sigma^1(x), \ldots, \sigma^r(x) \in T_x\mathcal{M}$$

for all $x \in \mathcal{M}$.

Part 2

Stochastic partial differential equations and invariant manifolds in embedded Hilbert spaces

Normed spaces with continuous embedding

- Let $(G, \| \cdot \|_G)$ and $(H, \| \cdot \|_H)$ be normed spaces.
- Then we call (G, H) normed spaces with *continuous embedding* if:
  1. We have $G \subset H$ as sets.
  2. The embedding operator $\mathrm{Id} : (G, \| \cdot \|_G) \to (H, \| \cdot \|_H)$ is continuous; that is, there is a constant $K > 0$ such that

$$\|x\|_H \leq K \|x\|_G \quad \text{for all } x \in G.$$

- In the sequel, we are interested in *continuous* mappings

$$A : (G, \| \cdot \|_G) \to (H, \| \cdot \|_H).$$

Stochastic partial differential equations

- Let (G, H) be continuously embedded separable Hilbert spaces.
- We consider the SPDE

$$\begin{cases} dY_t & = & L(Y_t)dt + A(Y_t)dW_t \\ Y_0 & = & y_0. \end{cases}$$

- Continuous coefficients $L : G \to H$ and $A : G \to \ell^2(H)$.
- Furthermore $W = (W^j)_{j \in \mathbb{N}}$ is an $\mathbb{R}^\infty$-Wiener process.
- A *martingale solution* Y is a G-valued adapted process on some stochastic basis $\mathbb{B}$ such that

$$Y_t = y_0 + \underbrace{\int_0^t L(Y_s)ds}_{\text{in } (H, \| \cdot \|_H)} + \underbrace{\int_0^t A(Y_s)dW_s}_{\text{in } (H, \| \cdot \|_H)}, \quad t \in \mathbb{R}_+.$$

Particular situations

- Semilinear SPDEs, where

$$G := \mathcal{D}(A),$$

endowed with the graph norm

$$\|y\|_G = \sqrt{\|y\|_H^2 + \|Ay\|_H^2}, \quad y \in G.$$

- SPDEs in Hermite Sobolev spaces with

$$G := \mathscr{S}_{p+1}(\mathbb{R}^d) \quad \text{and} \quad H := \mathscr{S}_p(\mathbb{R}^d).$$

The submanifold

- Recall the general SPDE

$$\begin{cases} dY_t & = & L(Y_t)dt + A(Y_t)dW_t \\ Y_0 & = & y_0. \end{cases} \quad (3)$$

- Let $\mathcal{M}$ be a (G, H)-submanifold of class C^2:
  1. $\mathcal{M}$ is a C^2-submanifold of H.
  2. We have $\mathcal{M} \subset G$.
  3. We have $\tau_H \cap \mathcal{M} = \tau_G \cap \mathcal{M}$.
- Let $\mathfrak{X}(\mathcal{M})$ be the space of all vector fields on $\mathcal{M}$; that is

$$A(y) \in T_y\mathcal{M} \quad \forall y \in \mathcal{M}.$$

The general invariance result

Theorem 2 – Bhaskaran & Tappe (2024)

$\mathcal{M}$ is locally invariant for the SPDE (3) if and only if

$$A^j|_{\mathcal{M}} \in \mathfrak{X}(\mathcal{M}), \quad j \in \mathbb{N}, \tag{4}$$

$$[L|_{\mathcal{M}}]_{\mathfrak{X}(\mathcal{M})} - \frac{1}{2} \sum_{j=1}^{\infty} [A^j|_{\mathcal{M}}, A^j|_{\mathcal{M}}]_{\mathcal{M}} = [0]_{\mathfrak{X}(\mathcal{M})}. \tag{5}$$

- The equation (5) is in the quotient space $\mathfrak{A}(\mathcal{M})/\mathfrak{X}(\mathcal{M})$.
- For $A, B \in \mathfrak{X}(\mathcal{M})$ the term $[A, B]_{\mathcal{M}}$ is locally given by

$$y \mapsto D^2\phi(x)\big(D\phi(x)^{-1}A(y), D\phi(x)^{-1}B(y)\big), \quad y \in U \cap \mathcal{M},$$

where $x := \phi^{-1}(y) \in V$.

Semilinear stochastic partial differential equations

Semilinear SPDEs

- We consider the H-valued semilinear SPDE

$$\begin{cases} dY_t &= (AY_t + \alpha(Y_t))dt + \sigma(Y_t)dW_t \\ Y_0 &= y_0. \end{cases} \quad (6)$$

- Here $A : H \supset D(A) \to H$ is a densely defined, closed operator.
- A could be the generator of a C_0-semigroup $(S_t)_{t \geq 0}$ on H.
- Furthermore $\alpha : H \to H$ and $\sigma : H \to \ell^2(H)$ are continuous.
- A *weak solution* Y is an H-valued adapted process on some stochastic basis $\mathbb{B}$ such that for all $\zeta \in D(A^*)$ we have

$$\langle \zeta, Y_t \rangle_H = \langle \zeta, y_0 \rangle_H + \int_0^t \big(\langle A^* \zeta, Y_s \rangle_H + \langle \zeta, \alpha(Y_s) \rangle_H \big) ds + \int_0^t \langle \zeta, \sigma(Y_s) \rangle_H dW_s, \quad t \in \mathbb{R}_+.$$

Continuous embeddings and the submanifold

- Consider the domain

$$G := \mathcal{D}(A),$$

endowed with the graph norm

$$\|y\|_G = \sqrt{\|y\|_H^2 + \|Ay\|_H^2}, \quad y \in G.$$

- (G, H) are continuously embedded separable Hilbert spaces.
- Moreover $A : (G, \| \cdot \|_G) \to (H, \| \cdot \|_H)$ is continuous.
- Let $\mathcal{M}$ be a C^2-submanifold of H.

The invariance result

Proposition 1 – Bhaskaran & Tappe (2024)

The following statements are equivalent:

1. $\mathcal{M}$ is locally invariant for the semilinear SPDE (6).
2. $\mathcal{M}$ is a (G, H)-submanifold, which is locally invariant for the continuously embedded SPDE (6).
3. $\mathcal{M}$ is a (G, H)-submanifold, and we have

$$\sigma^j|_{\mathcal{M}} \in \mathfrak{X}(\mathcal{M}), \quad j \in \mathbb{N}, \tag{7}$$

$$[(A + \alpha)|_{\mathcal{M}}]_{\mathfrak{X}(\mathcal{M})} - \frac{1}{2} \sum_{j=1}^{\infty} [\sigma^j|_{\mathcal{M}}, \sigma^j|_{\mathcal{M}}] = [0]_{\mathfrak{X}(\mathcal{M})}. \tag{8}$$

- If σ is of class C^1, then (8) is equivalent to

$$A|_{\mathcal{M}} + \alpha|_{\mathcal{M}} - \frac{1}{2} \sum_{j=1}^{\infty} D\sigma^j \cdot \sigma^j|_{\mathcal{M}} \in \mathfrak{X}(\mathcal{M}).$$

Remarks on the regularity

- Let $k, l \in \mathbb{N}$ be such that:
  1. $\mathcal{M}$ is a C^k-submanifold of H. ($k = 2$ admits Itô's formula)
  2. σ is of class C^l. ($l = 1$ admits Stratonovich term)
- In Filipović (2000) we have $k = 2$ and $l = 1$.
- In Nakayama (2004) we have $k = 1$ and $l = 1$.
- Here we have $k = 2$ and $l = 0$.
- In any case we have

$$k + l \geq 2.$$

Interplay between SPDEs and finite dimensional SDEs

Hermite Sobolev spaces

- Literature:
  1. Itô (1984).
  2. Kallianpur & Xiong (1995).
- Separable Hilbert spaces $(\mathscr{S}_p(\mathbb{R}^d))_{p\in\mathbb{R}}$ such that

$$\mathscr{S}(\mathbb{R}^d) \subset \mathscr{S}_p(\mathbb{R}^d) \subset \mathscr{S}'(\mathbb{R}^d) \quad \forall p \in \mathbb{R}.$$

- For $q \leq p$ we have the continuous embedding

$$\big(\mathscr{S}_p(\mathbb{R}^d), \mathscr{S}_q(\mathbb{R}^d)\big).$$

- For $q \leq 0 \leq p$ we have

$$\underbrace{\mathscr{S}(\mathbb{R}^d) \subset \mathscr{S}_p(\mathbb{R}^d) \subset \mathscr{S}_0(\mathbb{R}^d) = L^2(\mathbb{R}^d)}_{\text{functions}} \subset \underbrace{\mathscr{S}_q(\mathbb{R}^d) \subset \mathscr{S}'(\mathbb{R}^d)}_{\text{distributions}}.$$

Further properties

- For $k \in \mathbb{N}_0$ and $p > \frac{d}{4} + \frac{k}{2}$ we have the continuous embedding

$$\big(\mathscr{S}_p(\mathbb{R}^d), C_0^k(\mathbb{R}^d)\big).$$

- For each $p \in \mathbb{R}$ we obtain the dual pair

$$\big(\mathscr{S}_{-p}(\mathbb{R}^d), \mathscr{S}_p(\mathbb{R}^d), \langle \cdot, \cdot \rangle\big).$$

- Continuous linear operators

$$\partial_i : \mathscr{S}_{p+\frac{1}{2}}(\mathbb{R}^d) \to \mathscr{S}_p(\mathbb{R}^d).$$

Finite dimensional diffusions

- We consider the $\mathbb{R}^d$-valued SDE

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t \\ X_0 = x_0. \end{cases} \tag{9}$$

- Coefficients $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \ell^2(\mathbb{R}^d)$.
- Suppose that for some $q > \frac{d}{4}$ we have

$$b_i \in \mathscr{S}_q(\mathbb{R}^d) \quad \forall i = 1, \dots, d,$$
$$\sigma_i^j \in \mathscr{S}_q(\mathbb{R}^d) \quad \forall i = 1, \dots, d \quad \forall j \in \mathbb{N}.$$

- Let $\mathcal{N}$ be a C^2-submanifold of $\mathbb{R}^d$.

Definition of the SPDE

- We define the Hermite Sobolev spaces

$$G := \mathscr{S}_{-q}(\mathbb{R}^d) \quad \text{and} \quad H := \mathscr{S}_{-(q+1)}(\mathbb{R}^d).$$

- We consider the SPDE

$$\begin{cases} dY_t &= L(Y_t)dt + A(Y_t)dW_t \\ Y_0 &= y_0. \end{cases} \tag{10}$$

- Here $L : G \to H$ and $A : G \to \ell^2(H)$ are given by

$$L(y) := \frac{1}{2} \sum_{i,j=1}^{d} (\langle \sigma, y \rangle \langle \sigma, y \rangle^{\top})_{ij} \partial_{ij}^2 y - \sum_{i=1}^{d} \langle b_i, y \rangle \partial_i y,$$

$$A^j(y) := - \sum_{i=1}^{d} \langle \sigma_i^j, y \rangle \partial_i y, \quad j \in \mathbb{N}.$$

Definition of the submanifold

- We define the submanifold

$$\mathcal{M} := \{\delta_x : x \in \mathcal{N}\}.$$

- Here $\delta_x \in G$ is the *Dirac distribution*

$$\langle \delta_x, \varphi \rangle := \varphi(x) \quad \forall \varphi \in \mathscr{S}(\mathbb{R}^d).$$

Theorem 3 – Bhaskaran & Tappe (2024)

The following statements are equivalent:

1. $\mathcal{N}$ is locally invariant for the SDE (9).
2. $\mathcal{M}$ is locally invariant for the SPDE (10).

Finite dimensional diffusions

Submanifolds given by zeros of functions

- Recall the $\mathbb{R}^d$-valued SDE

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t \\ X_0 = x_0. \end{cases} \tag{11}$$

- Suppose $b_i \in \mathscr{S}_q(\mathbb{R}^d)$ and $\sigma_i^j \in \mathscr{S}_q(\mathbb{R}^d)$ for some $q > \frac{d}{4}$.
- We assume there is $f : \mathbb{R}^d \to \mathbb{R}^n$ such that

$$\mathcal{N} = \{x \in O : f(x) = 0\}, \quad \text{where } O \subset \mathbb{R}^d \text{ is open.}$$

- Here $m = d - n$, where $m = \dim \mathcal{N}$.
- We assume that $f_k \in \mathscr{S}_{q+1}(\mathbb{R}^d)$ for all $k = 1, \dots, n$.
- We also assume that $Df(x)\mathbb{R}^d = \mathbb{R}^n$ for all $x \in \mathcal{N}$.

The invariance result

Theorem 4 – Bhaskaran & Tappe (2024)

The following statements are equivalent:

1. $\mathcal{N}$ is locally invariant for the SDE (11).
2. For all $k = 1, \dots, n$ and all $x \in \mathcal{N}$ we have

$$\langle \sigma^j(x), \nabla f_k(x) \rangle = 0, \quad j \in \mathbb{N},$$

$$\langle b(x), \nabla f_k(x) \rangle + \frac{1}{2} \operatorname{tr}\big(\sigma(x) \sigma(x)^\top \mathbf{H}_{f_k}(x) \big) = 0.$$

- For the proof of (2) $\Rightarrow$ (1) we apply Theorem 3.

The unit sphere

- For $d \geq 2$ we consider the unit sphere

$$\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}.$$

Corollary 1

The following statements are equivalent:

1. $\mathbb{S}^{d-1}$ is (locally) invariant for the SDE (11).
2. For all $x \in \mathbb{S}^{d-1}$ we have

$$\langle \sigma^j(x), x \rangle = 0, \quad j \in \mathbb{N}, \tag{12}$$

$$\langle b(x), x \rangle + \frac{1}{2} \mathrm{tr}\big(\sigma(x)\sigma(x)^\top\big) = 0. \tag{13}$$

- For the proof consider $f(x) = \|x\|^2 - 1$.

Wiener process on the unit sphere

- We consider the $\mathbb{R}^d$-valued SDE

$$\begin{cases} dX_t = -\frac{d-1}{2} X_t dt + (\mathrm{Id} - X_t X_t^\top) dW_t \\ X_0 = x_0. \end{cases} \tag{14}$$

- Here W is an $\mathbb{R}^d$-valued Wiener process.

Example 1

The unit sphere $\mathbb{S}^{d-1}$ is invariant for the SDE (14).

- This is a consequence of Corollary 1.
- For example (12) is satisfied, because for all $x \in \mathbb{S}^{d-1}$ we have

$$(\mathrm{Id} - xx^\top)x = x - xx^\top x = x(1 - x^\top x) = x(1 - \|x\|^2) = 0.$$

Another example

- Consider the $\mathbb{R}^2$-valued SDE

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t \\ X_0 = x_0. \end{cases} \quad (15)$$

- Here W is an $\mathbb{R}$-valued Wiener process.
- The coefficients $b, \sigma : \mathbb{R}^2 \to \mathbb{R}^2$ are given by

$$b(x) := -\frac{1}{2}\lambda(x)^2 x,$$

$$\sigma(x) := \lambda(x)(-x_2, x_1)^\top.$$

- Here $\lambda : \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary continuous function.

Example 2

The unit sphere $\mathbb{S}^1$ is invariant for the SDE (15).

Related literature

- Papers about invariance for finite dimensional diffusions:
  1. Abi Jaber (2017); Abi Jaber, Bouchard & Illand (2019).
  2. Bardi & Goatin (1999); Bardi & Jensen (2002).
  3. Da Prato & Frankowska (2004).
- Choosing the mapping

$$\lambda : \mathbb{R}^2 \to \mathbb{R}, \quad \lambda(x) := |\arg(x)|^{\frac{1}{4}}$$

  the following statements are true:
  1. b and σ are continuous, but not locally Lipschitz on $\mathbb{S}^1$.
  2. σ is *not* of class C^1.
  3. $\sigma\sigma^\top$ is *not* of class C^1.

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