Kolmogorov equation associated with convective Brinkman-Forchheimer equations and its applications

International conference on stochastic calculus and applications to finance
IIT Madras, June 03-05, 2024

Sagar Gautam

Under the supervision of

Dr. Manil T. Mohan

Department of Mathematics

Indian Institute of Techonology Roorkee, India

June 3, 2024

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Transition functions

Definition

Let $\{X(t)\}_{t\geq 0}$ be a Markov process with values in $(E, \mathscr{E})$. The *transition probability function*[1] is a function $P(s, x, t, A)$ where $0 \leq s \leq t < \infty$, $x \in E$ and $A \in \mathscr{E}$ with the following properties:

1. For each $0 \leq s \leq t < \infty$ and each $x \in E$, $P(s, x, t, \cdot)$ is a probability measure on $\mathscr{E}$.
2. For each $0 \leq s \leq t < \infty$ and each $A \in \mathscr{E}$, $P(s, \cdot, t, A)$ is a $\mathscr{E}$-measurable function.
3. $\mathbb{P}[X(t) \in A | X(s)] = P(s, X(s), t, A)$, $\mathbb{P}$-a.s. for each $0 \leq s \leq t < \infty$ and $A \in \mathscr{E}$.
4. *Chapman-Kolmogorov equation.* For any $0 \leq s \leq u \leq t < \infty$, $x \in E$ and $A \in \mathscr{E}$, we have

$$P(s, x, t, A) = \int_E P(s, x, u, \mathrm{d}y) P(u, y, t, A). \tag{CKe}$$

[1] X. Mao, *Stochastic Differential Equations and Applications*, Elsevier, 2007.

Kolmogorov equation: An intuitive idea[2]

- Assume that the transition probabilities admits a density, say $p(s,x,t,y) \geq 0$. We will assume that $p(s,x,t,y)$ is smooth in s,x.
- If $s > 0$, then for a small $h > 0$, we have by Chapman-Kolmogorov equation,

$$p(s-h,x,t,y) = \int_{\mathbb{R}} p(s-h,x,s,z)p(s,z,t,y)\mathrm{d}z.$$

- Let us expand $p(s,z,t,y)$ around x as

$$p(s,z,t,y) = p(s,x,t,y) + (z-x)\frac{\partial}{\partial x}p(s,x,t,y) + \frac{1}{2}(z-x)^2\frac{\partial^2}{\partial x^2}p(s,x,t,y) + o(|z-x|^3).$$

- Assume that the limits $A(s,x) := \lim_{h\downarrow 0} \frac{1}{h}\int_{\mathbb{R}}(z-x)p(s-h,x,s,z)\mathrm{d}z$, and $B^2(s,x) := \lim_{h\downarrow 0} \frac{1}{h}\int_{\mathbb{R}}(z-x)^2 p(s-h,x,s,z)\mathrm{d}z$ exist.
- Then, for $t > s$, p fulfills the *backward equation*

$$-\frac{\partial}{\partial s}p(s,x,t,y) = A(s,x)\frac{\partial}{\partial x}p(s,x,t,y) + \frac{1}{2}B^2(s,x)\frac{\partial^2}{\partial x^2}p(s,x,t,y).$$

[2] A. Kolmogoroff, Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung, *Math. Ann.*, **104**(1) (1931), 415–458.

Kolmogorov equation: Analytic viewpoint[3]

- Consider the familiy of *Gaussian kernals*

$$p_t(x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2t}},\ t > 0, x \in \mathbb{R}^d.$$

- It is easy to see that p_t solves the heat equation:

$$\partial_t p_t = \frac{1}{2}\Delta p_t,$$

where Δ is the standard Laplacian in $\mathbb{R}^d$.

- From these kernels, we define the family of operators $\{P_t\}_{t\geq 0}$, for some suitable function $f : \mathbb{R}^d \to \mathbb{R}$, as

$$P_t f(x) := \int_{\mathbb{R}^d} f(y) p_t(x, y) \mathrm{d}y,\ t > 0,\ x \in \mathbb{R}^n,$$

with $p_t(x, y) = p_t(x - y)$, $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$.

- One may verify that $P_0 f = f$, where P_0 is the identity operator and $P_t \circ P_s = P_{t+s}$, for $t, s \geq 0$.

[3] D. Bakry, I. Gentil and M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators, **348**, Springer, 2014.*

Kolmogorov equation: Probabilistic view point[3]

- Let $\mathrm{C}_b(\mathbb{R}^d)$ denote the space of all bounded and uniformly continuous functions defined on $\mathbb{R}^d$ and it is a Banach space with respect to the norm $\|f\|_\infty = \sup_{x\in\mathbb{R}^d} |f(x)|$.
- Let $B(\cdot)$ is a d-dimensional Brownian motion in some probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
- It is worthwhile to note that $P_t f(x)$ can be expressed as

$$P_t f(x) = \int_{\mathbb{R}^d} f(y) \frac{1}{(2\pi t)^{\frac{d}{2}}} \exp\left\{ -\frac{|x-y|^2}{2t} \right\} \mathrm{d}y = \int_{\mathbb{R}^d} f(y) \mathcal{N}_{x,t\mathrm{I}}(\mathrm{d}y) = \mathbb{E} f(B(t))$$

- Clearly, $B(t)$ is solution to the following simplest Itô equation in $\mathbb{R}^d$:

$$\begin{cases} \mathrm{d}X(t) = \mathrm{d}B(t), \\ X(0) = 0. \end{cases}$$

Kolmogorov equation: Deterministic case

- We consider here the problem

$$\begin{cases} X'(t) = b(t, X(t)), \ t \in (s, T) \\ X(s) = x \in \mathbb{R}^d, \end{cases} \tag{KEd}$$

where $s \in [0, T)$ and $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$.

- We assume moreover b possesses a partial derivative $\mathrm{D}_x b$ which is continuous and bounded on $[0, T] \times \mathbb{R}^d$. As well known, under these assumptions problem (KEd) has a unique solution $X(\cdot) = X(\cdot, s, x) \in \mathrm{C}^1([0, T]; \mathbb{R}^d)$.
- Let us define the *transition evolution operator* for any $\varphi \in \mathscr{B}_b(\mathbb{R}^d)$ as

$$P_{s,t}\varphi(x) = \varphi(X(t, s, x)), \ x \in \mathbb{R}^d, \ s, t \in [0, T].$$

Proposition

For any $\varphi \in \mathrm{C}_b^1(\mathbb{R}^d)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{s,t}\varphi = P_{s,t}\mathscr{K}(t)\varphi \ \ \text{for} \ \ s, t \in (0, T),$$

where $\mathscr{K}(t)\varphi(x) = (b(t, x), \mathrm{D}_x\varphi(x))$, for $x \in \mathbb{R}^d$, $\varphi \in \mathrm{C}_b^1(\mathbb{R}^d)$ and $t \in (0, T)$.

Kolmogorov equation: Stochastic case[4]

- Let $B(\cdot)$ be a d-dimensional Brownian motion on probability space $(\Omega, \mathscr{F}_s, \mathbb{P})$. We are here concerned with the stochastic evolution equation in $\mathbb{R}^d$

$$\begin{cases} \mathrm{d}X(t) = b(t, X(t))\mathrm{d}t + \sigma(t, X(t))\mathrm{d}B(t), \\ X(s) = x, \end{cases} \tag{KEs}$$

where $0 \leq s < t \leq T$.

- In this case the *transition evolution operator* for any $\varphi \in \mathscr{B}_b(\mathbb{R}^d)$ is

$$P_{s,t}\varphi(x) = \mathbb{E}[\varphi(X(t, s, x))], \; x \in \mathbb{R}^d, \; s, t \in [0, T].$$

Proposition

Let $\varphi \in \mathrm{C}_b^2(\mathbb{R}^d)$. Then $P_{s,t}\varphi$ is differentiable with respect to t and we have

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{s,t}\varphi = P_{s,t}\mathscr{K}(t)\varphi, \quad \textit{for} \;\; t \geq 0, \tag{1}$$

where for all $t \in [0, T]$,

$$\mathscr{K}(s)\varphi(x) = \frac{1}{2}\,\mathrm{Tr}[\mathrm{D}_x^2\varphi(x)\sigma(s, x)\sigma^*(s, x)] + (b(s, x), \mathrm{D}_x\varphi(x)), \;\; \varphi \in \mathrm{C}_b^2(\mathbb{R}^d).$$

[4] G. D. Prato, *Introduction to Stochastic Analysis and Malliavin Calculus*, Third edition, Volume 13, Pisa, 2014.

Example[4]

- Consider the following parabolic equation[5] in $\mathbb{R}^d$:

$$\begin{cases} u_t(t,x) = \dfrac{1}{2}\,\mathrm{Tr}[\mathrm{Q}\mathrm{D}_x^2 u_{xx}(t,x)] + (\mathrm{A}x, \mathrm{D}_x u(t,x)), \;\; t>0, \\ u(0,x) = \varphi(x), \end{cases} \tag{2}$$

where $\mathrm{A}, \mathrm{Q} \in \mathcal{L}(\mathbb{R}^d)$, Q is symmetric and positive definite.

- The corresponding stochastic differential equation is

$$\begin{cases} dX(t) = \mathrm{A}X(t)\mathrm{d}t + \sqrt{\mathrm{Q}}\mathrm{d}B(t), \\ X(0) = x \in \mathbb{R}^d. \end{cases} \tag{3}$$

- The solution of (3) is given by variation of constants formula

$$X(t,x) = e^{t\mathrm{A}}x + \int_0^t e^{(t-s)\mathrm{A}}\sqrt{\mathrm{Q}}\mathrm{d}B(s), \quad t \geq 0.$$

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[4] G. D. Prato, *Introduction to Stochastic Analysis and Malliavin Calculus*, Third edition, Volume 13, Pisa, 2014.

[5] N. V. Krylov, *Lectures on Elliptic and Parabolic equations in Höder spaces*, AMS, Providence, 1996.

Example[4]

- Therefore, the law of $X(t,x)$ is given by

$$\mathscr{L}(X(t,x)) = \mathcal{N}_{e^{t\mathrm{A}}x,\mathrm{Q}_t},$$

where

$$\mathrm{Q}_t = \int_0^t e^{s\mathrm{A}}\mathrm{Q}e^{s\mathrm{A}^*}\mathrm{d}s,\ t \geq 0,$$

and A^* is the adjoint of A.

- Consequently, the transition semigroup P_t looks like

$$P_t\varphi(x) := \mathbb{E}[\varphi(X(t,x))] = \int_{\mathbb{R}^d} \varphi(y)\mathcal{N}_{e^{t\mathrm{A}}x,\mathrm{Q}_t}(\mathrm{d}y),$$

and so, if $\varphi \in \mathrm{C}_b^2(\mathbb{R}^d)$, the solution of (2) is given by

$$u(t,x) = P_t\varphi(x), \quad t \geq 0,\ x \in \mathbb{R}^d.$$

- If, in particular, $\det \mathrm{Q}_t > 0$, we have

$$u(t,x) = (2\pi)^{-\frac{d}{2}}[\det \mathrm{Q}_t]^{-\frac{1}{2}} \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}\left(\mathrm{Q}_t^{-1}(y-e^{t\mathrm{A}}x),(y-e^{t\mathrm{A}}x)\right)\right)\varphi(y)\mathrm{d}y.$$

Convective Brinkman-Forchheimer(CBF) equations

Let $\mathcal{O} \subset \mathbb{R}^2$ be a bounded domain with the smooth boundary $\partial\mathcal{O}$. The motion of the incompressible fluid governed by the CBF equations[6] for $(t,\xi) \in (0,T) \times \mathcal{O}$

$$\begin{cases} \dfrac{\partial \boldsymbol{y}}{\partial t} - \underbrace{\mu\Delta\boldsymbol{y}}_{\textit{diffusion}} + \underbrace{(\boldsymbol{y}\cdot\nabla)\boldsymbol{y}}_{\textit{convection}} + \underbrace{\alpha\boldsymbol{y} + \beta|\boldsymbol{y}|^{r-1}\boldsymbol{y}}_{\textit{damping}} + \nabla p = \boldsymbol{f}, \quad \text{in } \mathcal{O}\times(0,T), \\ \nabla\cdot\boldsymbol{y}(t,\xi) = 0, \quad \text{in } \mathcal{O}\times[0,T], \\ \boldsymbol{y}(t,\xi) = \boldsymbol{0}, \quad \text{on } \partial\mathcal{O}\times[0,T], \\ \boldsymbol{y}(0,\xi) = \boldsymbol{x}(\xi), \quad \text{in } \mathcal{O}, \\ \displaystyle\int_{\mathcal{O}} p(t,\xi)d\xi = 0, \quad \text{in } (0,T), \end{cases} \tag{CBF}$$

where $\boldsymbol{y}(t,\xi)$ represents the *velocity field* of the fluid particle at time t and position ξ, $p(t,\xi)$ represents the pressure, and $\boldsymbol{f}$ is an external forcing.

- The constant $\mu > 0$ is *Brinkman coefficient* (effective viscosity), and $\alpha, \beta > 0$ represent the *Darcy* and *Forchheimer coefficients*, respectively.
- The *absorption exponent* $r \in [1,\infty)$ and $r = 3$ is known as *critical exponent.*

[6] K. Kinra and M. T. Mohan, Random attractors and invariant measures for stochastic convective Brinkman-Forchheimer equations on 2D and 3D unbounded domains, *Discrete Contin. Dyn. Syst. Ser. B*, **29** (1), 2024.

Function spaces

- Let $\mathrm{C}_0^\infty(\mathcal{O};\mathbb{R}^2)$ denotes the space of all infinitely differentiable functions ($\mathbb{R}^2$-valued) with compact support in $\mathcal{O}\subset\mathbb{R}^2$.
- We define

$$
\begin{aligned}
\mathcal{V} &:= \{\boldsymbol{x}\in \mathrm{C}_0^\infty(\mathcal{O},\mathbb{R}^2) : \nabla\cdot\boldsymbol{x}=0\},\\
\mathbb{H} &:= \text{the closure of } \ \mathcal{V} \ \text{ in the Lebesgue space } \mathbb{L}^2(\mathcal{O})=\mathrm{L}^2(\mathcal{O};\mathbb{R}^2),\\
\mathbb{V} &:= \text{the closure of } \ \mathcal{V} \ \text{ in the Sobolev space } \mathbb{H}_0^1(\mathcal{O})=\mathrm{H}_0^1(\mathcal{O};\mathbb{R}^2),\\
\widetilde{\mathbb{L}}^p &:= \text{the closure of } \ \mathcal{V} \ \text{ in the Lebesgue space } \mathbb{L}^p(\mathcal{O})=\mathrm{L}^p(\mathcal{O};\mathbb{R}^2),
\end{aligned}
$$

  for $p\in(2,\infty)$.
- We characterize the spaces $\mathbb{H}$ and $\mathbb{V}$ with the norms

$$
\|\boldsymbol{x}\|_{\mathbb{H}}^2 := \int_{\mathcal{O}} |\boldsymbol{x}(\xi)|^2 \mathrm{d}\xi \ \text{ and } \ \|\boldsymbol{x}\|_{\mathbb{V}}^2 := \int_{\mathcal{O}} |\nabla\boldsymbol{x}(\xi)|^2 \mathrm{d}\xi
$$

  respectively, and $\|\boldsymbol{x}\|_{\widetilde{\mathbb{L}}^p}^p = \int_{\mathcal{O}} |\boldsymbol{x}(\xi)|^p \mathrm{d}\xi$, for $p\in[2,\infty)$.
- Let $(\cdot,\cdot)$ denotes the inner product in the Hilbert space $\mathbb{H}$ and $\langle\cdot,\cdot\rangle$ denotes the induced duality between the spaces $\mathbb{V}$ and its dual $\mathbb{V}'$ as well as $\widetilde{\mathbb{L}}^p$ and its dual $\widetilde{\mathbb{L}}^{p'}$, where $\frac{1}{p}+\frac{1}{p'}=1$.

Operators

Projection operator

Let $\mathcal{P} : \mathbb{L}^2(\mathcal{O}) \to \mathbb{H}$ be the *Helmholtz-Hodge orthogonal projection.*

Linear operator

We define the *Stokes operator* by $\mathrm{A}\boldsymbol{y} := -\mathcal{P}\Delta\boldsymbol{y}$, $\boldsymbol{y} \in \mathrm{D}(\mathrm{A}) := \mathbb{V} \cap \mathbb{H}^2(\mathcal{O})$.

Bilinear operator

- Let us define the trilinear form $b(\cdot,\cdot,\cdot) : \mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ by

$$b(\boldsymbol{y},\boldsymbol{z},\boldsymbol{w}) = \int_{\mathcal{O}} (\boldsymbol{y}(\xi)\cdot\nabla)\boldsymbol{z}(\xi)\cdot\boldsymbol{w}(\xi)d\xi = \sum_{i,j=1}^{2} \int_{\mathcal{O}} y_i(\xi)\frac{\partial z_j(\xi)}{\partial \xi_i} w_j(\xi)\mathrm{d}\xi.$$

- We also define the operator $\mathcal{B}(\cdot,\cdot) : \mathbb{V} \times \mathbb{V} \to \mathbb{V}'$ by $\langle \mathcal{B}(\boldsymbol{y},\boldsymbol{z}),\boldsymbol{w}\rangle := b(\boldsymbol{y},\boldsymbol{z},\boldsymbol{w})$.
- We denote $\mathcal{B}(\boldsymbol{y}) := \mathcal{B}(\boldsymbol{y},\boldsymbol{y}) = \mathcal{P}[(\boldsymbol{y}\cdot\nabla)\boldsymbol{y}]$.

Nonlinear operator

- Let us now consider the operator $\mathcal{C}(\boldsymbol{y}) := \mathcal{P}(|\boldsymbol{y}|^{r-1}\boldsymbol{y})$ for $\boldsymbol{y} \in \mathbb{V}$.
- The operator $\mathcal{C}(\cdot) : \mathbb{V} \to \mathbb{V}'$ is well-defined.

Stochastic convective Brinkman-Forchheimer equations

We consider the following stochastic convective Brinkman-Forchheimer (SCBF) equations perturbed by additive noise:

$$\begin{cases} \mathrm{d}\boldsymbol{Y}(t,\xi) - \mu\Delta\boldsymbol{Y}(t,\xi) + (\boldsymbol{Y}(t,\xi)\cdot\nabla)\boldsymbol{Y}(t,\xi) + \beta|\boldsymbol{Y}(t)|^{r-1}\boldsymbol{Y}(t,\xi) + \nabla p(t,\xi) \\ \qquad = \sqrt{\mathrm{Q}}\mathrm{dW}(t,\xi), \quad \text{in } \mathcal{O}\times(0,T), \\ \nabla\cdot\boldsymbol{Y}(t,\xi) = 0, \quad \text{in } \mathcal{O}\times[0,T], \\ \boldsymbol{Y}(t,\xi) = \boldsymbol{0}, \quad \text{on } \partial\mathcal{O}\times[0,T], \\ \boldsymbol{Y}(0,\xi) = \boldsymbol{y}(\xi), \quad \text{in } \mathcal{O}, \end{cases}$$

where the unknown $\boldsymbol{Y}(\cdot,\cdot)$ is a real valued process depending on $\xi\in\mathcal{O}$ and $\mathrm{W}(\cdot)$ is an $\mathbb{L}^2(\mathcal{O})$-valued Wiener process.

Abstract formulation of the stochastic system

Let us set $\boldsymbol{X}(t,\boldsymbol{x}) := \mathcal{P}\boldsymbol{Y}(t,\boldsymbol{y})$, $\boldsymbol{x} := \mathcal{P}\boldsymbol{y}$ and $\mathrm{W}(t) := \mathcal{P}\mathrm{W}(t)$. On projecting the SCBF, we get

$$\begin{cases} \mathrm{d}\boldsymbol{X}(t) + [\mu\mathrm{A}\boldsymbol{X}(t) + \mathcal{B}(\boldsymbol{X}(t)) + \beta\mathcal{C}(\boldsymbol{X}(t))]\mathrm{d}t = \sqrt{\mathrm{Q}}\mathrm{dW}(t), \ t\in(0,T), \\ \boldsymbol{X}(0) = \boldsymbol{x}, \end{cases} \tag{ASE}$$

where $\boldsymbol{x}\in\mathbb{H}$ and $\sqrt{\mathrm{Q}}\mathrm{dW}$ is a *colored noise* defined on a stochastic basis $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{0\le t\le T},\mathbb{P})$ with values in $\mathbb{H}$.

Assumptions

- Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space equipped with an increasing family of sub-sigma fields $\{\mathscr{F}_t\}_{0\leq t\leq T}$ of $\mathscr{F}$ satisfying the *usual conditions*.
- Let $\mathcal{L}(\mathbb{H},\mathbb{H})$ be the space of all bounded linear operators on $\mathbb{H}$. Let the covariance operator $\mathrm{Q} \in \mathcal{L}(\mathbb{H},\mathbb{H})$ be such that Q is positive, symmetric and trace class operator with $\ker \mathrm{Q} = \{\mathbf{0}\}$.
- We assume that there exists a complete orthonormal system $\{\boldsymbol{e}_k\}_{k\in\mathbb{N}}$ in $\mathbb{H}$ of the covariance operator Q and a bounded sequence $\{\mu_k\}_{k\in\mathbb{N}}$ of positive real numbers such that $\mathrm{Q}\boldsymbol{e}_k = \mu_k \boldsymbol{e}_k,\ k \in \mathbb{N}$. Here μ_k is an eigenvalue corresponding to the eigenfunction $\boldsymbol{e}_k$ such that following holds:

$$\mathrm{Tr}\,\mathrm{Q} = \sum_{k=1}^{\infty} \mu_k < \infty \quad \text{and} \quad \sqrt{\mathrm{Q}}\boldsymbol{y} = \sum_{k=1}^{\infty} \sqrt{\mu_k}(\boldsymbol{y}, \boldsymbol{e}_k)\boldsymbol{e}_k, \quad \text{for} \quad \boldsymbol{y} \in \mathbb{H}.$$

- We shall assume further that

$$\sqrt{\mathrm{Q}} \in \mathcal{L}(\mathbb{U}, \mathbb{H}),$$

where $\mathbb{U}$ is a Hilbert space, $\mathbb{H} \subset \mathbb{U}$ and the injection of $\mathbb{H}$ into $\mathbb{U}$ is Hilbert-Schmidt.

Example

- For $\varepsilon > 1$, one can take $\mathrm{Q} = \mathrm{A}^{-\varepsilon}$, $\{\mu_k\}_{k\in\mathbb{N}} = \{\lambda_k^{-\varepsilon}\}_{k\in\mathbb{N}}$ and $\mathbb{U} = \mathbb{V}_{-\varepsilon} = \mathrm{D}(\mathrm{A}^{-\frac{\varepsilon}{2}})$. Note that the asymptotic growth of λ_k are given by $\lambda_k \sim k$, for $k = 1, 2, \ldots$.
- Then, we calculate

$$\operatorname{Tr}\mathrm{Q} = \sum_{k=1}^{\infty}(\mathrm{Q}\boldsymbol{e}_k, \boldsymbol{e}_k) = \sum_{k=1}^{\infty}(\mathrm{A}^{-\varepsilon}\boldsymbol{e}_k, \boldsymbol{e}_k) = \sum_{k=1}^{\infty}\lambda_k^{-\varepsilon} \sim \sum_{k=1}^{\infty}\frac{1}{k^{\varepsilon}} < \infty,$$

  provided $\varepsilon > 1$.
- Furthermore the embedding $\mathbb{H} \hookrightarrow \mathbb{V}_{-\varepsilon}$ is Hilbert-Schmidt, that is, the map $\mathrm{J} : \mathbb{V}_{-\varepsilon} \to \mathbb{H}$ is a Hilbert-Schmidt operator. Indeed

$$\|\mathrm{J}\|^2_{\mathrm{L}_2(\mathbb{H},\mathbf{U})} = \sum_{k=1}^{\infty}\|\mathrm{J}\boldsymbol{e}_k\|^2_{\mathbf{U}} = \sum_{k=1}^{\infty}\|\boldsymbol{e}_k\|^2_{\mathbf{U}} = \sum_{k=1}^{\infty}(\mathrm{A}^{-\varepsilon}\boldsymbol{e}_k, \boldsymbol{e}_k) < \infty,$$

  provided $\varepsilon > 1$.
- Moreover, we calculate

$$\operatorname{Tr}(\mathrm{AQ}) = \sum_{k=1}^{\infty}((\mathrm{AQ})\boldsymbol{e}_k, \boldsymbol{e}_k)_{\mathbb{H}} = \sum_{k=1}^{\infty}(\mathrm{A}^{-\varepsilon+1}\boldsymbol{e}_k, \boldsymbol{e}_k) \sim \sum_{k=1}^{\infty}k^{-(\varepsilon-1)} < \infty,$$

  provided $\varepsilon > 2$.

Solution of SCBF equations

Global strong solution

Let $\boldsymbol{x} \in \mathbb{H}$ be given. An $\mathbb{H}$-valued $\{\mathscr{F}_t\}_{0\leq t\leq T}$-adapted stochastic process $\boldsymbol{X}(\cdot)$ is called a *strong solution* to the system (ASE) if the following conditions are satisfied:

(i) the process $\boldsymbol{X} \in \mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,T;\mathbb{H})\cap\mathrm{L}^2(0,T;\mathbb{V}))\cap\mathrm{L}^{r+1}(\Omega;\mathrm{L}^{r+1}(0,T;\widetilde{\mathbb{L}}^{r+1}))$ and $\boldsymbol{X}(\cdot)$ has a $\mathbb{V}$-valued modification, which is progressively measurable with continuous paths in $\mathbb{H}$ and $\boldsymbol{X} \in \mathrm{C}([0,T];\mathbb{H})\cap\mathrm{L}^2(0,T;\mathbb{V})$, $\mathbb{P}$-a.s.,

(ii) the following equality holds for every $t \in [0,T]$, as an element of $\mathbb{V}'$, $\mathbb{P}$-a.s.

$$\boldsymbol{X}(t) = \boldsymbol{X}_0 - \int_0^t [\mu \mathrm{A}\boldsymbol{X}(s) + \mathcal{B}(\boldsymbol{X}(s)) + \beta\mathcal{C}(\boldsymbol{X}(s))]\mathrm{d}s + \int_0^t \sqrt{\mathrm{Q}}\mathrm{dW}(s), \quad \text{(GS)}$$

(iii) the following Itô formula holds true for all $t \in [0,T]$, $\mathbb{P}$-a.s.:

$$\|\boldsymbol{X}(t)\|_{\mathbb{H}}^2 + 2\alpha\int_0^t \|\boldsymbol{X}(s)\|_{\mathbb{H}}^2\mathrm{d}s + 2\mu\int_0^t \|\boldsymbol{X}(s)\|_{\mathbb{V}}^2\mathrm{d}s + 2\beta\int_0^t \|\boldsymbol{X}(s)\|_{\widetilde{\mathbb{L}}^{r+1}}^{r+1}\mathrm{d}s$$
$$= \|\boldsymbol{x}\|_{\mathbb{H}}^2 + t\,\mathrm{Tr}(\mathrm{Q}) + 2\int_0^t (\sqrt{\mathrm{Q}}\mathrm{dW}(s), \boldsymbol{X}(s)).$$

Well-posedness to SCBF equations[6]

Theorem

Let $\boldsymbol{x} \in \mathbb{H}$ be given. For $r \in [1,3]$, under the aforementioned assumptions, there exists a pathwise unique strong solution $\boldsymbol{X}(\cdot)$ to the system (ASE)[7] such that

$$\boldsymbol{X} \in \mathrm{L}^2(\Omega; \mathrm{L}^\infty(0,T;\mathbb{H}) \cap \mathrm{L}^2(0,T;\mathbb{V})) \cap \mathrm{L}^{r+1}(\Omega; \mathrm{L}^{r+1}(0,T;\widetilde{\mathbb{L}}^{r+1})),$$

with $\mathbb{P}$-a.s., continuous trajectories in $\mathbb{H}$ and $\boldsymbol{X} \in \mathrm{C}([0,T];\mathbb{H}) \cap \mathrm{L}^2(0,T;\mathbb{V})$, $\mathbb{P}$-a.s. Moreover, for $\mathrm{Tr}(\mathrm{Q}) < \infty$, we have following energy estimate:

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\boldsymbol{X}(t)\|_{\mathbb{H}}^2 + 4\mu\int_0^T \|\nabla \boldsymbol{X}(t)\|_{\mathbb{H}}^2 \mathrm{d}t + 4\alpha \int_0^T \|\boldsymbol{X}(t)\|_{\mathbb{H}}^2 \mathrm{d}t + 4\beta\int_0^T \|\boldsymbol{X}(t)\|_{\widetilde{\mathbb{L}}^{r+1}}^{r+1}\mathrm{d}t\right]$$
$$\leq 2\left[\|\boldsymbol{x}\|_{\mathbb{H}}^2 + 7\,\mathrm{Tr}(\mathrm{Q})T\right].$$

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[6] K. Kinra and M. T. Mohan, Random attractors and invariant measures for stochastic convective Brinkman-Forchheimer equations on 2D and 3D unbounded domains, *Discrete Contin. Dyn. Syst. Ser. B*, **29** (1), 2024.

[7] M. T. Mohan, Stochastic convective Brinkman-Forchheimer equations, *Submitted*.

Existence of an invariant measure

- Applying infinite-dimensional Itô's formula to the process $\|\boldsymbol{X}(\cdot)\|_{\mathbb{H}}^2$, we obtain

$$\frac{2\mu}{t}\mathbb{E}\left[\int_0^t \|\boldsymbol{X}(s)\|_{\mathbb{V}}^2 \mathrm{d}s\right] \leq \frac{1}{t_0}\|\boldsymbol{x}\|_{\mathbb{H}}^2 + \mathrm{Tr}(\mathrm{Q}), \text{ for all } t > t_0.$$

- By applying Markov's inequlaity, we obtain

$$\lim_{r\to\infty}\sup_{t>t_0}\frac{1}{t}\int_0^t \mathbb{P}\{\|\boldsymbol{X}(s)\|_{\mathbb{V}} > r\}\mathrm{d}s \leq \lim_{r\to\infty}\sup_{t>t_0}\frac{1}{r^2}\mathbb{E}\left[\frac{1}{t}\int_0^t \|\boldsymbol{X}(s)\|_{\mathbb{V}}^2\mathrm{d}s\right] = 0. \quad \text{(T)}$$

- Let us set $\zeta_{t,\boldsymbol{x}}(\cdot) = \frac{1}{t}\int_0^t \lambda_{s,\boldsymbol{x}}(\cdot)\mathrm{d}s$, where $\lambda_{t,\boldsymbol{x}}(\Lambda) = \mathbb{P}\{\boldsymbol{X}(t,\boldsymbol{x}) \in \Lambda\},\ \Lambda \in \mathscr{B}(\mathbb{H})$, is the law of $\boldsymbol{X}(t,\boldsymbol{x})$ for each $\boldsymbol{x} \in \mathbb{H}$.
- From (T), the sequence of probability measures $\{\zeta_{t,\boldsymbol{x}}\}_{t>0}$ is tight and hence by the *Krylov-Bogoliubov theorem*[8] that there is an invariant measure η for the transition semigroup $\{P_t\}_{t\geq 0}$.

[8] G. D. Prato and J. Zabczyk, *Ergodicity for Infinite-Dimensional Systems*, London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1996.

Kolmogorov operator

- Let us denote by $P_t : \mathrm{C}_b(\mathbb{H}) \to \mathrm{C}_b(\mathbb{H})$, the transition semigroup

$$(P_t\psi)(\boldsymbol{x}) = \mathbb{E}[\psi(\boldsymbol{X}(t,\boldsymbol{x}))],\ \boldsymbol{x} \in \mathbb{H},\ t \geq 0,\ \psi \in \mathrm{C}_b(\mathbb{H}),$$

  where $\boldsymbol{X} = \boldsymbol{X}(t,\boldsymbol{x})$ is the unique strong solution of the SCBF system (ASE).
- Let us introduce the following space:

$$\mathscr{E}_{\mathrm{A}}(\mathbb{H}) := \text{ linspan } \{\varphi_h(\boldsymbol{x}) = e^{i(h,\boldsymbol{x})} : h \in \mathrm{D}(\mathrm{A})\},$$

  and on $\mathscr{E}_{\mathrm{A}}(\mathbb{H})$, the following Kolmogorov differential operator:

$$(N_0\psi)(\boldsymbol{x}) = \frac{1}{2}\operatorname{Tr}\left[\mathrm{Q}\mathrm{D}^2_{\boldsymbol{x}}\psi(\boldsymbol{x})\right] - (\mu\mathrm{A}\boldsymbol{x} + \alpha\boldsymbol{x} + \mathcal{B}(\boldsymbol{x}) + \beta\mathcal{C}(\boldsymbol{x}), \mathrm{D}_{\boldsymbol{x}}\psi(\boldsymbol{x})), \quad \text{(KO)}$$

  for all $\psi \in \mathscr{E}_{\mathrm{A}}(\mathbb{H})$.
- It is well known that the transition semigroup P_t associated with (ASE), can be uniquely extended to a strongly continuous semigroup of contractions on $\mathbb{L}^2(\mathbb{H};\eta)$, still denoted by itself, since $\mathrm{C}_b(\mathbb{H})$ is dense in $\mathbb{L}^2(\mathbb{H};\eta)$.
- Let us denote by $N : \mathrm{D}(N) \subset \mathbb{L}^2(\mathbb{H};\eta) \to \mathbb{L}^2(\mathbb{H};\eta)$ as the infinitesimal generator of P_t.

Estimates on derivatives

Lemma

Let us write $\xi^h(t,\boldsymbol{x}) := \mathrm{D}_{\boldsymbol{x}}\boldsymbol{X}(t,\boldsymbol{x})h$, *for all* $\boldsymbol{x}, h \in \mathbb{H}$ *and assume that* $\mu^3\lambda_1 + 2\alpha\mu^2 > \max\{4\|\mathrm{Q}\|_{\mathcal{L}(\mathbb{H})}, 2\,\mathrm{Tr}(\mathrm{Q})\}$. *Then, we have*

$$\mathbb{E}\Big[\|\xi^h(t,\boldsymbol{x})\|_{\mathbb{H}}^2\Big] \leq \|h\|_{\mathbb{H}}^2 e^{\frac{2}{\mu^2}\|\boldsymbol{x}\|_{\mathbb{H}}^2} e^{-\left(\mu\lambda_1 + 2\alpha - \frac{2}{\mu^2}\mathrm{Tr}(\mathrm{Q})\right)t}, \tag{Ee3}$$

for all $t \in [0,T]$.

Existence and uniqueness of invariant measure

There exists an invariant measure η for P_t. Furthermore, if the condition

$$\mu^3\lambda_1 + 2\alpha\mu^2 > \max\{4\|\mathrm{Q}\|_{\mathcal{L}(\mathbb{H})}, 2\,\mathrm{Tr}(\mathrm{Q})\},$$

holds true, then the invariant measure is unique.

Approximations

- Now our aim is to study the infinitesimal generator N of P_t. Once again, we consider the Kolmogorov operator (KO).
- Applying Itô's formula, it follows easily that $N\psi = N_0\psi$, for all $\psi \in \mathscr{E}_{\mathrm{A}}(\mathbb{H})$. Our main goal is to show that $\mathscr{E}_{\mathrm{A}}(\mathbb{H})$ is the core of N.
- In order to do this, we need to find an estimate for

$$\int_{\mathbb{H}} \|\mathrm{A}^{\delta}\boldsymbol{x}\|_{\mathbb{H}}^{2m-2}\|\mathrm{A}^{\delta+\frac{1}{2}}\boldsymbol{x}\|_{\mathbb{H}}^{2}\eta(\mathrm{d}\boldsymbol{x}), \quad \text{where} \quad \delta > 0 \quad \text{and} \quad m \in \mathbb{N}.$$

- We first approximate (ASE) by the regular equations:

$$\begin{cases} \mathrm{dX}_{\varepsilon}(t) + [\mu \mathrm{AX}_{\varepsilon}(t) + \alpha \mathrm{X}_{\varepsilon}(t) + \mathcal{B}_{\varepsilon}(\mathrm{X}_{\varepsilon}(t)) + \beta\mathcal{C}_{\varepsilon}(\mathrm{X}_{\varepsilon}(t))]\mathrm{d}t = \sqrt{\mathrm{Q}}\mathrm{dW}(t), \ t \geq 0, \\ \mathrm{X}_{\varepsilon}(0) = \boldsymbol{x} \in \mathbb{H}, \end{cases} \tag{AE}$$

where

$$\mathcal{B}_{\varepsilon}(\boldsymbol{x}) = \begin{cases} \mathcal{B}(\boldsymbol{x}) & \text{if } \|\boldsymbol{x}\|_{\mathbb{V}} \leq \varepsilon^{-1}, \\ \varepsilon^{-2}\|\boldsymbol{x}\|_{\mathbb{V}}^{-2}\mathcal{B}(\boldsymbol{x}) & \text{if } \|\boldsymbol{x}\|_{\mathbb{V}} > \varepsilon^{-1}. \end{cases}$$

and

$$\mathcal{C}_{\varepsilon}(\boldsymbol{x}) = \begin{cases} \mathcal{C}(\boldsymbol{x}) & \text{if } \|\boldsymbol{x}\|_{\mathbb{V}} \leq \varepsilon^{-1}, \\ \varepsilon^{-(r+1)}\|\boldsymbol{x}\|_{\mathbb{V}}^{-(r+1)}\mathcal{C}(\boldsymbol{x}) & \text{if } \|\boldsymbol{x}\|_{\mathbb{V}} > \varepsilon^{-1}. \end{cases}$$

Main results

Result-1

Let us assume that $\mathrm{Tr}(\mathrm{QA}^{2\delta}) < +\infty$, for some $\delta \in (\frac{1}{4}, \frac{1}{2})$. Then, there are some positive constants γ_i, for $i = 1, 2, 3$ depending on m such that if $\mu > C\vartheta(\delta, \mathrm{Q})$, then following estimate holds for all $\varepsilon > 0$ and for all $m \in \mathbb{N}$:

$$
\begin{aligned}
&k_1 \int_{\mathbb{H}} e^{\lambda \|\boldsymbol{x}\|_{\mathbb{H}}^2} \|\mathrm{A}^{\delta}\boldsymbol{x}\|_{\mathbb{H}}^{2m} \Big(1 + \lambda\|\boldsymbol{x}\|_{\mathbb{H}}^2 + \|\boldsymbol{x}\|_{\widetilde{\mathbb{L}}^{r+1}}^{r+1}\Big) \nu_{\varepsilon}(\mathrm{d}\boldsymbol{x}) \\
&+ k_2 \int_{\mathbb{H}} e^{\lambda \|\boldsymbol{x}\|_{\mathbb{H}}^2} \|\mathrm{A}^{\delta}\boldsymbol{x}\|_{\mathbb{H}}^{2(m-1)} \|\mathrm{A}^{\delta+\frac{1}{2}}\boldsymbol{x}\|_{\mathbb{H}}^2 \nu_{\varepsilon}(\mathrm{d}\boldsymbol{x}) \\
&+ k_3 \int_{\mathbb{H}} e^{\lambda \|\boldsymbol{x}\|_{\mathbb{H}}^2} \|\mathrm{A}^{\delta}\boldsymbol{x}\|_{\mathbb{H}}^{2m} \|\boldsymbol{x}\|_{\mathbb{V}}^2 \nu_{\varepsilon}(\mathrm{d}\boldsymbol{x}) \leq \gamma_1.
\end{aligned}
$$

Result-2

Assume that $\mathrm{Tr}(\mathrm{AQ}) < +\infty$. Then, we have

$$
\int_{\mathbb{H}} \|\mathrm{A}\boldsymbol{x}\|_{\mathbb{H}}^2 \eta(\mathrm{d}\boldsymbol{x}) \leq C(\|\mathrm{Q}\|_{\mathcal{L}(\mathbb{H})}, \mathrm{Tr}(\mathrm{Q}), \beta, \mu).
$$

Infinitesimal generator of transition semigroup

- We say that a linear operator $\mathscr{A} : \mathrm{D}(\mathscr{A}) \subset \mathcal{H} \to \mathcal{H}$ in a Hilbert space $\mathcal{H}$ is *dissipative* if

$$\|\varphi\|_{\mathcal{H}} \leq \frac{1}{\lambda}\|\lambda\varphi - \mathscr{A}\varphi\|_{\mathcal{H}} \text{ for all } \varphi \in \mathrm{D}(\mathscr{A}),\ \lambda > 0.$$

- Any dissipative operator is closable. The dissipative operator $\mathscr{A}$ is called *m-dissipative* if the range of $\lambda\mathrm{I} - \mathscr{A}$ coincides with $\mathcal{H}$ for some (and consequently for any) $\lambda > 0$.

Lumer-Phillips theorem[6]

Let $\mathscr{A} : \mathrm{D}(\mathscr{A}) \subset \mathcal{H} \to \mathcal{H}$ be a dissipative operator in the Hilbert space $\mathcal{H}$ such that $\mathrm{D}(\mathscr{A})$ is dense in $\mathcal{H}$. Assume that for some $\lambda > 0$, the range of $\lambda\mathrm{I} - \mathscr{A}$ is dense in $\mathcal{H}$. Then the closure of $\mathscr{A}$ is m-dissipative.

Result 3: Essential m-dissipativity

Assume that the condition $\mu > C\vartheta(\delta, \mathrm{Q})$, holds true and that $\mathrm{Tr}(\mathrm{A}^{\rho}\mathrm{Q}) < +\infty$, for some $\rho > 2/3$. Then N_0 is dissipative in $\mathbb{L}^2(\mathbb{H}; \eta)$ and its closure $\overline{N}_0$ in $\mathbb{L}^2(\mathbb{H}; \eta)$ coincides with the infinitesimal generator N of P_t in $\mathbb{L}^2(\mathbb{H}; \eta)$.

[6] G. D. Prato, *Kolmogorov Equations for Stochastic PDEs*, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser Verlag, Basel, 2004.

The "Carre du Champ's" identity

- The following identity is straightforward:

$$N_0(\varphi^2) = 2\varphi N_0\varphi + \|\sqrt{Q}\mathrm{D}_{\boldsymbol{x}}\varphi\|^2_{\mathbb{H}} \ \text{ for all } \varphi \in \mathscr{E}_{\mathrm{A}}(\mathbb{H}).$$

- By exploiting the invariance of η and integrating the aforementioned identity with respect to η over $\mathbb{H}$, we obtain

$$\int_{\mathbb{H}} N_0\varphi(\boldsymbol{x})\varphi(\boldsymbol{x})\eta(\mathrm{d}\boldsymbol{x}) = -\frac{1}{2}\int_{\mathbb{H}} \|\sqrt{\mathrm{Q}}\mathrm{D}_{\boldsymbol{x}}\varphi(\boldsymbol{x})\|^2_{\mathbb{H}}\eta(\mathrm{d}\boldsymbol{x}).$$

- Let us now discuss the infinitesimal generator N of the semigroup $\{\mathrm{P}_t\}_{t\geq 0}$. We endow the domain $\mathrm{D}(N)$ of N with the following graph norm:

$$\|\varphi\|^2_{\mathrm{D}(N)} = \|\varphi\|^2_{\mathbb{L}^2(\mathbb{H};\eta)} + \|N\varphi\|^2_{\mathbb{L}^2(\mathbb{H};\eta)}, \ \varphi \in \mathrm{D}(N).$$

Lemma

The operator $\mathrm{Q}^{\frac{1}{2}}\mathrm{D}_{\boldsymbol{x}}\varphi$ defined in $\mathscr{E}_{\mathrm{A}}(\mathbb{H})$, is uniquely extendible to a linear bounded operator from $\mathrm{D}(N)$ into $\mathbb{L}^2(\mathbb{H},\eta;\mathbb{H})$. The extension is still denoted by $\mathrm{Q}^{\frac{1}{2}}\mathrm{D}_{\boldsymbol{x}}\varphi$. Moreover, we have the following "Carre du Champ's" identity:

$$\int_{\mathbb{H}} N\varphi(\boldsymbol{x})\varphi(\boldsymbol{x})\eta(\mathrm{d}\boldsymbol{x}) = -\frac{1}{2}\int_{\mathbb{H}} \|\sqrt{\mathrm{Q}}\mathrm{D}_{\boldsymbol{x}}\varphi(\boldsymbol{x})\|^2_{\mathbb{H}}\eta(\mathrm{d}\boldsymbol{x}) \ \textit{ for all } \ \varphi \in \mathrm{D}(N)$$

and $\|\mathrm{Q}^{\frac{1}{2}}\mathrm{D}_{\boldsymbol{x}}\varphi\|_{\mathbb{L}^2(\mathbb{H},\eta;\mathbb{H})} \leq \|\varphi\|_{\mathrm{D}(N)}$ *for all* $\varphi \in \mathrm{D}(N)$.

Applications: Infinite horizon problem

- We consider an infinite horizon problem described by the state equation for incompressible 2D stochastic convective Brinkman-Forchheimer fluids for $t > 0$:

$$\begin{cases} \mathrm{d}\boldsymbol{X}(t) + [\mu \mathrm{A}\boldsymbol{X}(t) + \alpha \boldsymbol{X}(t) + \mathcal{B}(\boldsymbol{X}(t)) + \beta \mathcal{C}(\boldsymbol{X}(t))]\mathrm{d}t = \sqrt{\mathrm{Q}}\mathrm{U}(t)\mathrm{d}t + \sqrt{\mathrm{Q}}\mathrm{d}\mathrm{W}(t), \\ \boldsymbol{X}(0) = \boldsymbol{x}. \end{cases} \tag{CE}$$

- We consider a cost functional of the form

$$J_\infty(\mathrm{U}) = \mathbb{E}\left\{\int_0^\infty e^{-\lambda t}[f(\boldsymbol{X}(t,\boldsymbol{x};\mathrm{U}(t))) + h(\mathrm{U}(t))]\mathrm{d}t\right\},$$

over all adapted square integrable controls U, where f and h are real valued funtions on $\mathbb{H}$ and $\lambda > 0$ is a discount factor.
- We define admissible class of control process

$$\mathcal{U}_{\mathrm{ad}} := \left\{\mathrm{U}(\cdot) \in \mathrm{L}^2(\Omega, \mathrm{L}^2(0,\infty;\mathbb{H})) : \|\mathrm{U}(t)\|_{\mathbb{H}} \leq R,\ \mathbb{P}\text{-a.s. and } \mathrm{U}(\cdot) \text{ is } \mathscr{F}_t \text{ adapted} \right\},$$

where $R > 0$ is fixed, corresponding to fixed reference probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
- We again call $(\boldsymbol{X}(\cdot), \mathrm{U}(\cdot))$ an admissible control pair if $\mathrm{U}(\cdot)$ is an $\mathscr{F}_t$-adapted process with values in $\mathbb{H}$ and $\boldsymbol{X}(\cdot)$ is a weak solution to (CE) corresponding to $\mathrm{U}(\cdot)$.
- We define the value function $\mathcal{V} : \mathbb{H} \to \mathbb{R}$ corresponding to cost functional, as

$$\mathcal{V}(\boldsymbol{x}) := \inf_{\mathrm{U}(\cdot)\in\mathcal{U}_{\mathrm{ad}}} J_\infty(\mathrm{U}) = \inf_{\mathrm{U}\in\mathcal{U}_{\mathrm{ad}}} \mathbb{E}\left\{\int_0^\infty e^{-\lambda t}[f(\boldsymbol{X}(t,\boldsymbol{x};\mathrm{U}(\cdot))) + h(\mathrm{U}(t))]\mathrm{d}t\right\}.$$

Applications: Infinite horizon problem

- We consider the following infinite dimensional second order stationary Hamilton-Jacobi Bellman equation related to the stochastic optimal problem (KO):

$$\lambda\varphi(\boldsymbol{x}) - \frac{1}{2}\operatorname{Tr}\left[\mathrm{Q}\mathrm{D}_{\boldsymbol{x}}^2\varphi(\boldsymbol{x})\right] \tag{HJB}$$

$$+ (\mu\mathrm{A}\boldsymbol{x} + \alpha\boldsymbol{x} + \mathcal{B}(\boldsymbol{x}) + \beta\mathcal{C}(\boldsymbol{x}), \mathrm{D}_{\boldsymbol{x}}\varphi(\boldsymbol{x})) + g(\mathrm{Q}^{1/2}\mathrm{D}_{\boldsymbol{x}}\varphi(\boldsymbol{x})) = f(\boldsymbol{x}),$$

  where $\lambda > 0$, $f \in \mathbb{L}^2(\mathbb{H};\eta)$ and the Hamiltonian $g : \mathbb{H} \to \mathbb{R}$ is Lipschitz continuous.
- Moreover, g is defined as the Legendre transform of the convex function $h : \mathbb{H} \to \mathbb{R}$:

$$g(\boldsymbol{x}) = \sup_{\boldsymbol{y}\in\mathbb{H}}\{(\boldsymbol{x},\boldsymbol{y}) - h(\boldsymbol{y})\}, \ \boldsymbol{x} \in \mathbb{H}.$$

Example

1. Let $h(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x}\|_{\mathbb{H}}^2$ for $\boldsymbol{x} \in \mathbb{H}$. Then, the Hamiltonian is given by $g(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x}\|_{\mathbb{H}}^2$.
2. Let $R > 0$ be given and

$$h(\boldsymbol{x}) = \begin{cases} \frac{1}{2}\|\boldsymbol{x}\|_{\mathbb{H}}^2, & \text{if } \|\boldsymbol{x}\|_{\mathbb{H}} \le R, \\ +\infty, & \text{if } \|\boldsymbol{x}\|_{\mathbb{H}} > R, \end{cases}$$

   Then, the Hamiltonian $g(\cdot)$ is explicitly given by

$$g(\boldsymbol{x}) = \begin{cases} \frac{1}{2}\|\boldsymbol{x}\|_{\mathbb{H}}^2, & \text{if } \|\boldsymbol{x}\|_{\mathbb{H}} \le R, \\ R\|\boldsymbol{x}\|_{\mathbb{H}} - \frac{R^2}{2}, & \text{if } \|\boldsymbol{x}\|_{\mathbb{H}} > R. \end{cases}$$

Optimal stopping problem

- Let $\boldsymbol{X}(\cdot)$ be the process associated with the following SCBF equations:

$$\begin{cases} \mathrm{d}\mathrm{X}(r) + [\mu \mathrm{A}\mathrm{X}(r) + \alpha \mathrm{X}(r) + \mathcal{B}(\mathrm{X}(r)) + \beta \mathcal{C}(\mathrm{X}(r))]\mathrm{d}r = \sqrt{\mathrm{Q}}\mathrm{d}\mathrm{W}(r), \ r \geq t, \\ \mathrm{X}(t) = \boldsymbol{x}. \end{cases}$$

- Let us define the value function of an optimal stopping problem associated with SCBF equations as

$$\varphi(t, \boldsymbol{x}) := \inf_{\tau \in \mathfrak{M}} \left\{ \mathbb{E}\left[\int_t^\tau \mathrm{F}(s, \mathrm{X}(s))\mathrm{d}s \right] + \mathbb{E}[\mathrm{G}(\mathrm{X}(\tau))] \right\}, \qquad \text{(VF)}$$

where $\mathfrak{M}$ is the family of all $\{\mathscr{F}_t\}_{t\geq 0}$ stopping times such that $\tau \in [t, T]$ $\mathbb{P}$-a.s., $\mathrm{F} : (0, \infty) \times \mathbb{H} \to \mathbb{R}$ and $\mathrm{G} : \mathbb{H} \to \mathbb{R}$ are given functions.

Optimal stopping problem

- It can be seen that the value function φ defined by (VF) (after a suitable change of time variable) is formally the solution to the following variational inequality:

$$\begin{cases} \dfrac{\partial \varphi}{\partial t}(t,\boldsymbol{x}) - \dfrac{1}{2}\operatorname{Tr}\big[\mathrm{Q}\mathrm{D}_{\boldsymbol{x}}^2\varphi(t,\boldsymbol{x})\big] + (\mu \mathrm{A}\boldsymbol{x} + \alpha\boldsymbol{x} + \mathcal{B}(\boldsymbol{x}) + \beta\mathcal{C}(\boldsymbol{x}), \mathrm{D}_{\boldsymbol{x}}\varphi(t,\boldsymbol{x})) \leq \mathrm{F}(t,\boldsymbol{x}), \\ \quad \text{for all } \ t \geq 0, \ \boldsymbol{x} \in \mathrm{D}(\mathrm{A}), \ \varphi(t,\boldsymbol{x}) \leq \mathrm{G}(\boldsymbol{x}), \ \ \text{for all } \ t \geq 0, \ \boldsymbol{x} \in \mathbb{H}, \\ \dfrac{\partial \varphi}{\partial t}(t,\boldsymbol{x}) - \dfrac{1}{2}\operatorname{Tr}\big[\mathrm{Q}\mathrm{D}_{\boldsymbol{x}}^2\varphi(t,\boldsymbol{x})\big] + (\mu \mathrm{A}\boldsymbol{x} + \alpha\boldsymbol{x} + \mathcal{B}(\boldsymbol{x}) + \beta\mathcal{C}(\boldsymbol{x}), \mathrm{D}_{\boldsymbol{x}}\varphi(t,\boldsymbol{x})) = \mathrm{F}(t,\boldsymbol{x}), \\ \quad \text{in } \ \{\boldsymbol{x} : \varphi(t,\boldsymbol{x}) < \mathrm{G}(\boldsymbol{x})\}, \ \varphi(0,\boldsymbol{x}) = \varphi_0(\boldsymbol{x}), \ \boldsymbol{x} \in \mathbb{H}. \end{cases} \tag{OP}$$

- Let us define the closed convex subset of $\mathbb{L}^2(\mathbb{H};\eta)$ as

$$K = \big\{\varphi \in \mathbb{L}^2(\mathbb{H};\eta) : \varphi \leq \mathrm{G}, \ \eta\text{ -a.e.}\big\}.$$

- We are going to study the existence and uniqueness result for the problem (OP) which can be viewed as a nonlinear equation of the form:

$$\begin{cases} \dfrac{\mathrm{d}\varphi(t)}{\mathrm{d}t} - N\varphi(t) + N_K\varphi(t) \ni \mathrm{F}(t), \ t \in (0,T), \\ \varphi(0) = \varphi_0, \end{cases}$$

where $\varphi_0 \in \mathbb{L}^2(\mathbb{H};\eta)$ and $\mathrm{L}^2([0,T];\mathbb{L}^2(\mathbb{H};\eta))$ are given.

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Thank you for your kind attention!!