Properties of the gradient squared of the discrete Gaussian free field

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June 4, 2024

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Abelian sandpile

The height-one field

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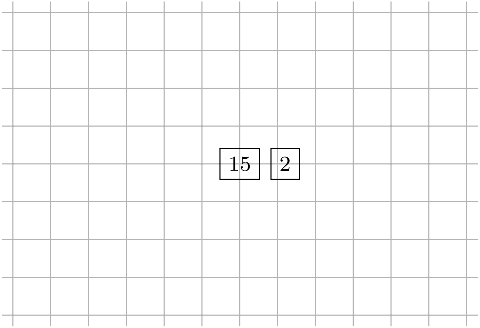
This Markov chain has a unique stationary measure $\mathbb{P}$. We look at

Definition (Height-one field)

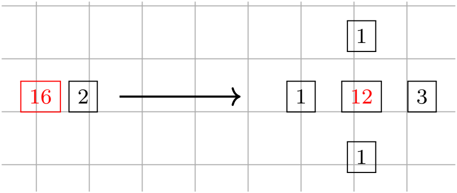
$h_\Lambda(x) := \mathbf{1}_{\{s(x)=1\}}$ under $\mathbb{P}$

An example

$$s(x) = 15\,\delta_{x=(0,\,0)} + 2\,\delta_{x=(1,0)}$$



An example



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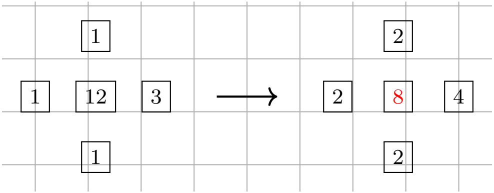

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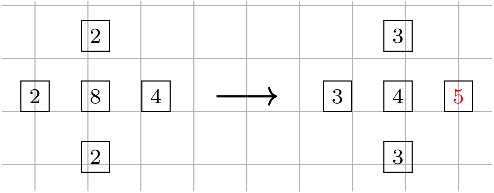

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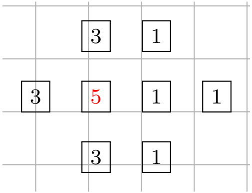

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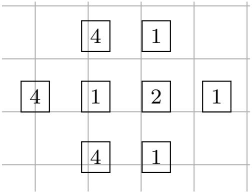


Stable configuration!

Going larger

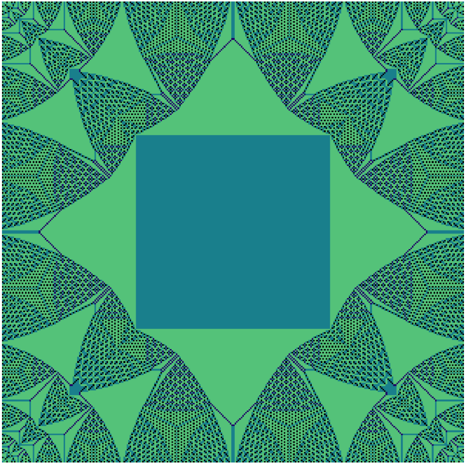
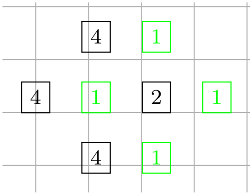


Figure: Sandpile configuration on a 300×300 box.

The height one field



Joint cumulants

Joint cumulants κ for r. v.'s $X_1, \ldots, X_n$ are defined by

$$E\left[\prod_{i=1}^{n} X_i\right] = \sum_{\pi \text{ partition of } \{1, \ldots, n\}} \prod_{B \in \pi} \kappa(X_i : i \in B)$$

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We are going to study

$$\kappa(h(x_1), \ldots, h(x_n))$$

Question

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- Regardless of the microscopic details of the model, what is the "driving force" behind height one?

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Ingredients

- Let $U \subset \mathbb{R}^2$ be smooth connected bounded and $\Lambda := U_\epsilon := {}^{U}/_{\epsilon} \cap \mathbb{Z}^2$
- Let
$$U \ni u \mapsto u_\epsilon = \lfloor {}^{u}/_{\epsilon} \rfloor \in U_\epsilon$$
- Let $g_U(\cdot, \cdot)$ be the harmonic Green's function on U with Dirichlet boundary conditions

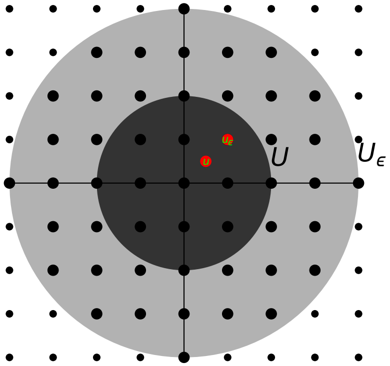

Figure: $U = B(0,1), U_\epsilon = B(0,2) \cap \mathbb{Z}^2$, $\epsilon = 1/2, u = (1/2, 1/2), u_\epsilon = (1,1)$

Theorem (Dürre (2009))

Theorem 2 (*Scaling Limit for the Height One Joint Cumulants*). *Let* V *be as in Theorem* 1 *and suppose* $|V| \geq 2$. *Then as* $\epsilon \to 0$ *the rescaled joint cumulant* $\epsilon^{-2|V|}\kappa\left(h_{U_\epsilon}(v_\epsilon) : v \in V\right)$ *converges to*

$$\kappa_U(v : v \in V) := -C^{|V|} \sum_{\sigma \in S_{\mathrm{cycl}}(V)} \sum_{(k^v)_{v \in V} \in \{x,y\}^V} \prod_{v \in V} \partial^{(1)}_{k^v} \partial^{(2)}_{k^{\sigma(v)}} g_U\left(v, \sigma(v)\right).$$

Here $C := (2/\pi) - (4/\pi^2)$. *That is, if we write* $\kappa_U(v) := 0$ *for all* $v \in V$, *then*

$$\lim_{\epsilon \to 0} \epsilon^{-2|V|} \mathbb{E}\left[\prod_{v \in V} \left(h_{U_\epsilon}(v_\epsilon) - \mathbb{E}[h_{U_\epsilon}(v_\epsilon)]\right)\right] = \sum_{\Pi \in \Pi(V)} \prod_{B \in \Pi} \kappa_U(v : v \in B).$$

Let Ψ be a Gaussian free field with 0-boundary conditions on U:

Definition (GFF)

Ψ is the centered Gaussian random distribution with

$$\mathbb{E}[\Psi(x)\Psi(y)] = g_U(x, y), \quad x \neq y \in U.$$

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We investigated this conjecture

Grad squared DGFF

Definition (DGFF)

Let $(\Gamma_\epsilon(v) : v \in U_\epsilon)$ be the discrete GFF on U_ϵ:

$$\mathbb{E}[\Gamma_\epsilon(v)] = 0, \quad \mathbb{E}[\Gamma_\epsilon(v)\Gamma_\epsilon(u)] = G_{U_\epsilon}(u, v)$$

where $G_{U_\epsilon}(\cdot, \cdot)$ is the discrete harmonic Green's function with Dirichlet b.c.

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We will work in $d \geq 2$ ($d = 1$: manual calculations)

Grad squared DGFF

Covariances

Call $[d] := \{1, \ldots, d\}$.

$$\mathsf{E}\left[\Phi_\epsilon(x_\epsilon)\Phi_\epsilon(y_\epsilon)\right] = 2\sum_{i,j\in[d]}\left(\nabla_i^{(1)}\nabla_j^{(2)}G_{U_\epsilon}(x_\epsilon,\, y_\epsilon)\right)^2$$

Main results

Convergence of cumulants

Theorem (Cipriani, Hazra, Rapoport, Ruszel 2023)

Let $\mathcal{E}$ be the set of coordinate vectors of $\mathbb{R}^d$. Let $\{x^{(1)}, \ldots, x^{(k)}\} \subset U$. Let $S^0_{\mathrm{cycl}}(B)$ be the set of cyclic permutations of a set B without fixed points. If $x^{(i)} \neq x^{(j)}$ for all $i \neq j$, then

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$$\lim_{\epsilon \to 0} \epsilon^{-dk} \kappa\Big(\Phi_\epsilon\big(x_\epsilon^{(j)}\big) : j \in [k]\Big) = 2^{k-1} \sum_{\sigma \in S^0_{\mathrm{cycl}}([k])} \sum_{\eta : [k] \to \mathcal{E}} \prod_{j=1}^{k} \partial^{(1)}_{\eta(j)} \partial^{(2)}_{\eta(\sigma(j))} g_U\left(x_\epsilon^{(j)}, x_\epsilon^{(\sigma(j))}\right)$$

In $d = 2$ the limit is conformally covariant with scale dimension 2

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Corollary

$$-2 \lim_{\epsilon\to 0} \epsilon^{-2k} \kappa\Big(\frac{C}{2}\Phi_\epsilon(x_\epsilon^{(j)}) : j \in [k]\Big) = \lim_{\epsilon\to 0} \epsilon^{-2k} \kappa\Big(h_{U_\epsilon}(x_\epsilon^{(j)}) : j \in [k]\Big)$$

Main results

Convergence as random distribution

Consider for $f \in C_c^\infty(U)$, $U \subset \mathbb{R}^d$,

$$\langle \Phi_\epsilon, f \rangle = \int_U \Phi_\epsilon(x_\epsilon)\, f(x)\, \mathrm{d}x.$$

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Theorem (C, Hazra, Rapoport, Ruszel 2022)

$$\chi^{-1/2}\, \epsilon^{-d/2}\, \Phi_\epsilon \xrightarrow{d} \textit{white noise on } U,$$

in $\mathcal{C}^\alpha_{\mathrm{loc}}(U)$ *for any* $\alpha < -d/2$*, and the constant* χ *is*

$$\chi := 2 \sum_{v \in \mathbb{Z}^d} \sum_{i,j \in [d]} \left(\nabla_i^{(1)} \nabla_j^{(2)} G_0(0, v) \right)^2 \in (0, +\infty)$$

where $G_0(\cdot, \cdot)$ *is the* $\begin{cases} \textit{infinite-volume discrete Green's function} & \textit{in } d \geq 3 \\ \textit{potential kernel} & \textit{in } d = 2 \end{cases}$

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✓

Recap

What we have so far

Random distribution Scaling $\epsilon^{-d/2}$ and limit are the same as height-one field

Cumulants Scaling ϵ^{-d} and limit are the same as height-one field up to sign

Proofs

Why white noise?

- Finite susceptibility ($\iff \chi \in (0, +\infty)$) suggests CLT-type rescaling and WN convergence Bauerschmidt et al. (2014), Newman (1980)...

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- Kassel–Wu (2013) derive Gaussian fluctuations for models related to the spanning tree measure (reprove Dürre)
- We are not able to apply K–W's results directly, but this hints at a universality class of models related to the spanning tree measure via the transfer current matrix $T(\cdot, \cdot)$

$$\mathbb{E}\left[\nabla_i \Gamma_\epsilon(v) \nabla_j \Gamma_\epsilon(u)\right] = T\big((v, v+e_i), (u, u+e_j)\big)$$

Proofs
Useful facts: white noise

For $v, w \in U_\epsilon$ "away from the boundary"

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For v, $w \in U_\epsilon$ "away from the boundary"

- $\mathsf{E}\left[\Phi_\epsilon(v)\Phi_\epsilon(w)\right] \leq c \cdot \begin{cases} |v-w|^{-2d} & v \neq w \ , \\ 1 & v = w \ . \end{cases}$

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- $\left|\nabla_i^{(1)}\nabla_j^{(2)} G_{U_\epsilon}(v,w) - \nabla_i^{(1)}\nabla_j^{(2)} G_0(v,w)\right| \leq c\,\epsilon^d\ ,$

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   - the limiting covariance structure is the $L^2(U)$ inner product

Proofs

Why these cumulants?

To answer, we need to look at the proof first...

Proofs

Useful facts: cumulants

- $T\big((x_\epsilon, x_\epsilon + e), (y_\epsilon,\, y_\epsilon + e')\big) = \epsilon^d \, \mathrm{d}g_U|_{(x,y)}(e,\, e') + o(\epsilon^d)$ (Kassel–Wu, 2013).

Proofs

k-point functions

We derive cumulants from k-point functions:

$$\mathbb{E}\left[\prod_{j=1}^{k} \Phi_\epsilon\left(x_\epsilon^{(j)}\right)\right] = \sum_{\pi \in \Pi([k])} \prod_{B \in \pi} \kappa\left(\Phi_\epsilon\left(x_\epsilon^{(j)}\right) : j \in B\right)$$

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In the proof we (loosely) obtain that the k-point function is

$$\sum_{\gamma \text{ FD on } [2k]} \prod_{\left((x_\epsilon^{(j)}, x_\epsilon^{(j)}+e), (x_\epsilon^{(m)}, x_\epsilon^{(m)}+e')\right) \in \gamma} \epsilon^{-d} T\left((x_\epsilon^{(j)}, x_\epsilon^{(j)}+e), (x_\epsilon^{(m)}, x_\epsilon^{(m)}+e')\right)$$

Proofs

Cumulants: another viewpoint

In the proof we (loosely) obtain that the k-point function is

$$\sum_{\gamma \text{ FD on } [2k]} \prod_{\left((x_\epsilon^{(j)}, x_\epsilon^{(j)}+e),(x_\epsilon^{(m)},\, x_\epsilon^{(m)}+e')\right)\in\gamma} \underbrace{\epsilon^{-d} T\Big((x_\epsilon^{(j)}, x_\epsilon^{(j)}+e)\Big), (x_\epsilon^{(m)},\, x_\epsilon^{(m)}+e')\Big)}_{\approx\, \partial_e \partial_{e'} g_U(x^{(j)}, x^{(m)})}$$

Fermionic (or Grassmannian) calculus

Definition (Grassmanian variables)

Let $\{\xi_i, \bar{\xi}_i : i \in \Lambda\}$ be symbols that satisfy for all i, j

$$\xi_i \xi_j = -\xi_j \xi_i, \quad \xi_i \bar{\xi}_j = -\bar{\xi}_j \xi_i, \quad \bar{\xi}_i \bar{\xi}_j = -\bar{\xi}_j \bar{\xi}_i$$

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Example

Used in physics to model Fermi–Dirac statistics (opposed to Bose–Einstein statistics)

Fermionic Gaussian free field

Definition (fGFF)

For every function F of $\{\boldsymbol{\xi}, \bar{\boldsymbol{\xi}}\} = \{\xi_i,\, \bar{\xi}_i : i \in \Lambda\}$ the expectation of F under the fGFF is defined as

$$[F]_{fGFF} = \int_{Berezin} \mathrm{d}\bar{\boldsymbol{\xi}} \mathrm{d}\boldsymbol{\xi}\, \mathrm{e}^{(\boldsymbol{\xi}, -\Delta_\Lambda \bar{\boldsymbol{\xi}})}\, F.$$

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$$[F]_{DGFF} \propto \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{\varphi}\, \mathrm{e}^{\frac{1}{4d}(\boldsymbol{\varphi},\, -\Delta_\Lambda \boldsymbol{\varphi})}\, F$$

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$$[1]_{fGFF} = \int_{Ber} \mathrm{d}\bar{\boldsymbol{\xi}}\mathrm{d}\boldsymbol{\xi}\, \mathrm{e}^{(\boldsymbol{\xi}, -\Delta_\Lambda \bar{\boldsymbol{\xi}})} = \det(-\Delta_\Lambda)$$

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$$\int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{\varphi}\, \mathrm{e}^{\frac{1}{4d}(\boldsymbol{\varphi},\, -\Delta_\Lambda \boldsymbol{\varphi})} \propto (\det(-\Delta_\Lambda))^{-1/2}$$

“Fermionic gradient squared”

For $v \in \Lambda = U_\epsilon$

$$X_v = \frac{1}{2d} \sum_{e \ni v \text{ edges}} \zeta(e)$$

$$\zeta(e) = \left(\xi_v - \xi_u\right)\left(\bar{\xi}_v - \bar{\xi}_u\right), \quad e = \{v,\, u\}$$

Theorem (CCRR, 2023)

$$\lim_{\epsilon \to 0} \epsilon^{-2n} \kappa\left(h_{U_\epsilon}\left(v_\epsilon^{(1)}\right), \dots, h_{U_\epsilon}\left(v_\epsilon^{(n)}\right)\right)$$
$$\lim_{\epsilon \to 0} \epsilon^{-2n} \kappa\left(-CX_{v_\epsilon^{(1)}}, \dots, -CX_{v_\epsilon^{(n)}}\right).$$

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- As a random distribution the limit is WN as height-one field: common decay of correlations
- The cumulants have the same limit as in the height-one field (up to sign) and conformal covariance property
- the Fermionic free field kind of identifies the height-one field and gives an alternative the gradient free field squared.

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Thank you!