A random string among Poissonian obstacles

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Random String

$\mathbf{u}(t,x) \in \mathbb{R}^d$, $x \in [0, J]$, $t \in [0, T]$, with periodic boundary conditions

$$\begin{aligned} \partial_t \mathbf{u}(t,x) &= \partial_x^2 \mathbf{u}(t,x) + \dot{\mathbf{W}}(t,x), \\ \mathbf{u}(0,x) &= \mathbf{u}_0(x) \end{aligned}$$

$\dot{\mathbf{W}}$ is a vector of independent white noises: Distribution valued centred Gaussian process

$$\mathbb{E}\left(\dot{W}(h)\dot{W}(g)\right) = \int h(t,x)g(t,x)\,dt\,dx$$

Heat kernel $G : [0, T] \times [0, J] \to \mathbb{R}$

$$\partial_t G(t,x) = \frac{1}{2}\,\partial_x^2 G(t,x), \qquad G(0,x) = \delta(x).$$

Random string solves

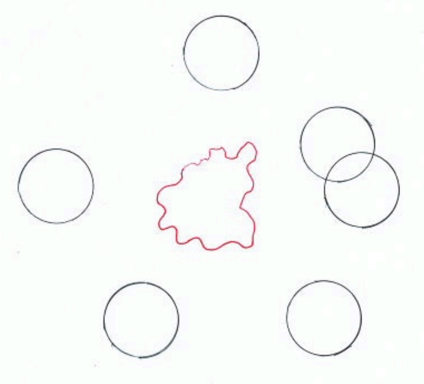
$$\mathbf{u}(t,x) = G_t * \mathbf{u}_0(x) + \mathbf{N}(0,t;x),$$

where $G_t * u_0(x)$ is the *determinstic* term and the *noise* term

$$\mathbf{N}(0,t;x) = \int_0^t \int_0^J G_{t-s}(x,y)\mathbf{W}(ds,dy).$$

Obstacles

$\boldsymbol{\eta}$: Poisson point process on $\mathbb{R}^d$ with intensity ν. (independent of string)
Closed balls of radius a around each point.



Survival: No part of the string hits any obstacle up to time T
Question: What is the probability of survival of the string?

Point process $\boldsymbol{\eta} = \sum_{i \geq 1} \delta_{\boldsymbol{\xi}_i}$
$\mathsf{H} : \mathbb{R}^d \to [0, \infty]$ is a compactly supported measurable function

$$\mathsf{V}(\mathbf{z}, \boldsymbol{\eta}) = \sum_{i \geq 1} \mathsf{H}(\mathbf{z} - \boldsymbol{\xi}_i),$$

Hard obstacles: $\mathsf{H} = \infty \cdot \mathbf{1}_{\mathcal{B}(\mathbf{0},a)}$
Soft obstacles: H does not take the value ∞

- Averaged Partition function

$$S_T = \mathbb{E}\left[\exp\left(-\int_0^T \int_0^J \mathsf{V}\left(\mathbf{u}(s,x), \boldsymbol{\eta}\right) dxds\right)\right]$$

  (averaging over $\dot{\mathbf{W}}$ and $\boldsymbol{\eta}$)
- For hard obstacles, S_T is the probability of survival up to time T
- Averaged Path Measure

$$\mathbb{Q}_T(d\omega) = \frac{\mathbb{E}_{\boldsymbol{\eta}}\left[\exp\left(-\int_0^t \int_0^J \mathsf{V}\Big(\mathbf{u}(s,x), \boldsymbol{\eta}\Big) dsdx\right)\right] \mathbb{P}_{\dot{\mathbf{W}}}(d\omega)}{S_T}.$$

  (measure on path space of the string after averaging over $\boldsymbol{\eta}$)

Brownian Motion among Poissonian obstacles

- The analogous partition function $\mathcal{S}_T = \mathbb{E}\left[\exp\left(-\int_0^T \mathrm{V}\left(B_s, \boldsymbol{\eta}\right) ds\right)\right]$
- $\mathcal{S}_T = \mathbb{E} v_{\boldsymbol{\eta}}(t, 0)$, where

$$\partial_t v_{\boldsymbol{\eta}}(t,x) = \Delta v_{\boldsymbol{\eta}}(t,x) - \mathrm{V}(x,\boldsymbol{\eta}) v_{\boldsymbol{\eta}}(t,x)$$
$$v_{\boldsymbol{\eta}}(t,0) = 1$$

- $v_{\boldsymbol{\eta}}(t,0)$ is in fact the *quenched* partition function.
- For hard obstacles, a Poissonian calculation shows

$$\mathcal{S}_T = \mathbb{E}\exp(-\nu|\chi_T(a)|),$$

where $\chi_T(a) = \cup_{0\le t\le T}\mathcal{B}(B_t, a)$ is the Weiner sausage
- For hard and soft obstacles

$$\mathcal{S}_T = \exp\left(-C T^{\frac{d}{d+2}}(1+o(1))\right) \text{ as } T \to \infty$$

- Under the averaged measure the paths have fluctuations of order $T^{\frac{1}{d+2}}$

Theorem (Hard obstacles)

Let $d \geq 2$, $J \geq 1$ and $H = \infty \cdot \mathbf{1}_{\mathcal{B}(0,a)}$. There are constants $C_0, C_1, \cdots, C_4$ independent of T, J such that for $T \geq C_0 J^{2+\frac{d}{2}}$

$$C_1 \exp\left(-C_2 \left(\frac{T}{J}\right)^{\frac{d}{d+2}}\right) \leq S_T \leq C_3 \exp\left(-\frac{C_4}{1+|\log J|}\left(\frac{T}{J^2}\right)^{\frac{d}{d+2}}\right).$$

Theorem (Soft obstacles)

Let $d \geq 2$, $J \geq 1$ and $H(x) \geq \mathcal{C} \cdot \mathbf{1}_{\mathcal{B}(0,a)}$. Fix $\beta > 0$. There are constants $C_0, C_1, \cdots, C_4$ independent of T, J such that for $T \geq C_0 J^{2+\frac{d}{2}}$

$$C_1 \exp\left(-C_2 \left(\frac{T}{J}\right)^{\frac{d}{d+2}}\right) \leq S_T \leq C_3 \exp\left(-\frac{C_4}{J^{3+\beta}\left(1+|\log J|\right)}\left(\frac{T}{J^2}\right)^{\frac{d}{d+2}}\right).$$

- The exponent of T is the same as in the case of Brownian Motion
- For hard obstacles

$$S_T = \mathbb{E} \exp\left(-\nu \left|\Gamma_T^J(a)\right|\right),$$

where $\Gamma_T^J(a) = \cup_{\substack{0\leq s\leq T\\ 0\leq x\leq J}} \mathcal{B}(\mathbf{u}(s,x), a)$

- Computing an upper bound for S_T amounts to computing lower bounds for $|\Gamma_T^J(a)|$.
- The upper bound actually holds for all $T > 0$ while the lower bound needs $T \geq C_0 J^{2+\frac{d}{2}}$.
- The constants are independent of T, J but depend on $\nu, a, \mathcal{C}, \beta$.

Broader context

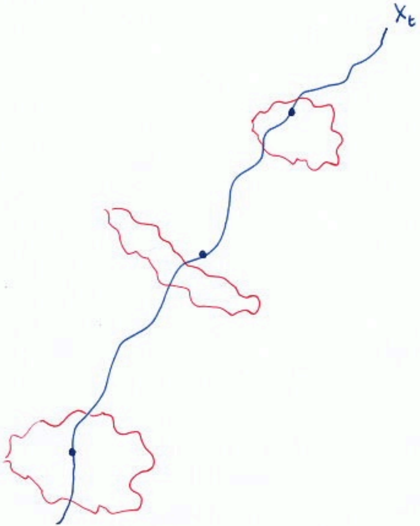
- In many statistical mechanics models one has a set of configurations $\omega \in \Omega$ that the physical system can take. Assume $|\Omega| < \infty$.
- Hamiltonian $H : \Omega \to \mathbb{R}$, $\quad H(\omega)$ is the *energy* of the configuration ω
- $\mu = (\mu(\omega))_{\omega\in\Omega}$ probability distribution on Ω
- Minimizing *entropy* $\sum_\omega \mu(\omega) \log \mu(\omega)$ subject to *average energy* $\sum_\omega \mu(\omega) H(\omega) = U$ (fixed), we obtain

$$\mu_\beta(\omega) = \frac{e^{-\beta H(\omega)}}{\sum_\omega e^{-\beta H(\omega)}}, \qquad \text{(Gibbs distribution)}$$

  where $\beta = \beta(\omega)$ is the *inverse temperature*, uniquely determined by $\sum_\omega \mu(\omega) H(\omega) = U$.
- The *partition function* $Z(\beta) = \sum_\omega e^{-\beta H(\omega)}$ contains a lot of information about the model.
- Many famous discrete models: Ising Model, Directed Polymers in Random Environment, Hard Core Lattice Gas, Potts Models
- In our model the energy function is given by $\int_0^T \int_0^J \mathrm{V}(\mathbf{u}(s,x), \boldsymbol{\eta}) dxds$.

(Assume from now on $J = 1$)

$$\mathbf{X}_t = \int_0^1 \mathbf{u}(t,x)\,dx, \text{ (Center of Mass) }, \qquad \mathbf{R}_t = \sup_{x \in [0,1]} |\mathbf{u}(t,x) - \mathbf{X}_t|, \text{ (Radius)}.$$



Lemma

(i) $\mathbf{X}_t$ *is a Brownian motion. (ii)* $\mathbf{X}_t$ *and* $\mathbf{R}_t$ *are independent.*

$$\mathbf{u}(t,x) = G_t * \mathbf{u}_0(x) + \int_0^t \int_0^1 G_{t-s}(x,y)\mathbf{W}(ds,dy)$$

$$\mathbf{X}_t = \int_0^1 \mathbf{u}_0(x)dx + \int_0^t \mathbf{W}(dyds)$$

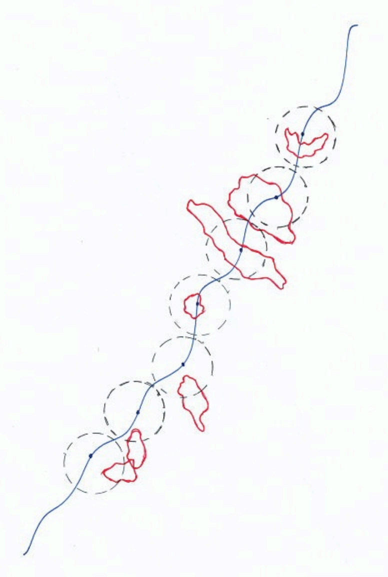
$$\mathbf{u}(t,x) - \mathbf{X}_t = \int_0^1 [G(t,x-y)-1]\,\mathbf{u}_0(y)dy + \int_{[0,t]\times[0,1]} [G(t-s,x-y)-1]\,\mathbf{W}(dsdy)$$

$$\mathrm{Cov}(\mathbf{X}_t, \mathbf{u}(t,x)-\mathbf{X}_t) = \int_0^t \int_0^1 [G(t-s,x-y)-1]\; dyds = 0.$$

Lower bound

- Consider the event A that there is a large *obstacle-free* ball of radius $\alpha + a$ around the origin, and the string lies within this ball up to time T.
- $$\mathbb{P}\left(\sup_{\substack{s\le T\\ x\in[0,1]}} |\mathbf{u}(s,x)| \le \frac{\alpha}{2}\right) \ge \mathbb{P}\left(\sup_{s\le T}|\mathbf{R}_s| \le \frac{\alpha}{4}\right)\mathbb{P}\left(\sup_{s\le T}|\mathbf{X}_s| \le \frac{\alpha}{4}\right)$$
  $$\ge \exp\left(-C\frac{T}{\alpha^2}\right)$$
  (For large T the fluctuations of $\mathbf{R}$ are *small* compared to that of $\mathbf{X}$)
- $$\mathbb{P}(\text{ no obstacles in ball of radius } \alpha + a \text{ around } \mathbf{0}) = \exp\left(-C\nu(\alpha + a)^d\right).$$
- $S_T \ge \mathbb{P}(A) \ge \exp\left(-C\frac{T}{\alpha^2}\right)\cdot \exp\left(-C\nu(\alpha+a)^d\right)$, optimize, $\alpha \approx T^{\frac{1}{d+2}}$.

Upper bound



Upper bound

- With high probability there are at least $T^{\frac{d}{d+2}}$ many time points $\tau_1 < \tau_2 < \cdots \leq T$ separated by at least L (large) such that the balls of radius Λ around X_{τ_i} are disjoint.
- For large t

$$\begin{aligned} \mathbf{u}(t,x) &= \int_0^1 G_t(x,y)\mathbf{u}_0(y)dy + \mathbf{N}(0,t;x) \\ &\approx \int_0^1 \mathbf{u}_0(y)dy + \mathbf{N}(0,t;x) \\ &= \mathbf{X}_0 + \mathbf{N}(0,t;x) \end{aligned}$$

- On the other hand

$$\mathbf{X}_t = \mathbf{X}_0 + \int_0^1 \mathbf{N}(0,t;y)dy$$

- For $\mathbf{f} : [0,1] \to \mathbb{R}^d$

$$\mathcal{R}(\mathbf{f}) := \sup_{x,y\in[0,1]} |\mathbf{f}(x) - \mathbf{f}(y)|, \qquad \mathscr{S}(\mathbf{f}) := \bigcup_{x\in[0,1]} \mathcal{B}(\mathbf{f}(x), a)$$

- For large t

$$\mathbb{P}\left(\mathcal{R}\left(\mathbf{N}(0,t;\cdot)\right) \le \frac{\Lambda}{2},\ \ |\mathscr{S}\left(\mathbf{N}(0,t;\cdot)\right)| \ge a^{d-\frac{3}{2}}\right) \ge \frac{1}{2}$$

- It is enough to show above for the spatial process

$$\begin{aligned} \mathbf{N}(0,t;\cdot) - \mathbf{N}(0,t;0) &\overset{d}{=} \mathbf{N}(-t,0;\cdot) - \mathbf{N}(-t,0;0) \\ &\approx \mathbf{N}(-\infty,0;\cdot) - \mathbf{N}(-\infty,0;0) \end{aligned}$$

- Therefore for large t, with probability at least $\frac{1}{2}$

$$|\mathbf{u}(t,x) - \mathbf{X}_t| \le \Lambda, \quad \text{and} \quad |\mathscr{S}(\mathbf{u}(t,\cdot))| \ge a^{d-\frac{3}{2}}. \qquad (*)$$

Upper bound

- We count the number of times τ_i at which (*) holds.
- If the events were i.i.d. then standard large deviation theory will guarantee that with high prob. a positive fraction of the $T^{\frac{d}{d+2}}$ balls around $\mathbf{X}_{\tau_i}$ would contain *spatial* sausages of volume $a^{d-\frac{3}{2}}$ around them, all of which are disjoint.
- However these are NOT independent.
- $$\mathbf{u}(t,\cdot) = G_{t-s} * \mathbf{u}(s,\cdot) + \mathbf{N}(s,t,\cdot)$$

  If $t-s$ large the first term becomes approximately constant. (see next slide)
- Recall that the τ_i are spaced L (large) apart. With high probability one can get a subsequence τ_{i_k} (of $O(T^{\frac{d}{d+2}})$ many τ_i's) such that

$$\begin{aligned} u\left(\tau_{i_{k+1}},\cdot\right) &= G_{\tau_{i_{k+1}}-\tau_{i_k}} * \mathbf{u}\left(\tau_{i_k},\cdot\right) + \mathbf{N}\left(\tau_{i_k},\tau_{i_{k+1}};\cdot\right) \\ &\approx \int_0^1 \mathbf{u}\left(\tau_{i_k},y\right)dy + \mathbf{N}\left(\tau_{i_k},\tau_{i_{k+1}};\cdot\right) \end{aligned}$$

- $\mathcal{R}\left(\mathbf{N}\left(\tau_{i_k},\tau_{i_{k+1}};\cdot\right)\right)$ and $\mathscr{S}\left(\mathbf{N}\left(\tau_{i_k},\tau_{i_{k+1}};\cdot\right)\right)$ are independent of the past

Lemma

For $\mathbf{f} : [0,1] \to \mathbb{R}^d$

$$\mathcal{R}(G_t * \mathbf{f}) \leq 4de^{-4\pi^2 t}\mathcal{R}(\mathbf{f}).$$

Proof.

We consider one component f of $\mathbf{f}$. Let $f(x) = \sum_k a_k e^{i2\pi kx}$.

$$\begin{aligned}
\sup_{x,y} |(G_t * f)(x) - (G_t * f)(y)| &= \sup_{x,y} \left| \sum_{k\neq 0} e^{-4\pi^2 k^2 t} a_k \left[e^{i2\pi kx} - e^{i2\pi ky} \right] \right| \\
&\leq 4e^{-4\pi^2 t} \left(\sum_{k\neq 0} a_k^2 \right)^{1/2}
\end{aligned}$$

Since $\int_0^1 f(x)dx$ is the zeroth Fourier coefficient of f we have

$$\left(\sum_{k\neq 0} a_k^2 \right)^{1/2} = \left\| f - \int_0^1 f(x)dx \right\|_2 \leq \left\| \mathbf{f} - \int_0^1 \mathbf{f}(x)dx \right\|_2 \leq \mathcal{R}(\mathbf{f})$$

□

THANK YOU