Existence and uniqueness of weak solutions for the generalized stochastic Navier-Stokes-Voigt equations

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Outline of Presentation

Introduction

- The motivation of this work comes from the article[1], where the author discussed the existence theory of stochastic power-law fluids.
- The major goal of this talk is to discuss the following:
  - The existence of a probabilistic weak solution to the generalized stochastic Navier-Stokes-Voigt (GSNSV) equations perturbed by a multiplicative Gaussian noise.
  - The pathwise uniqueness of solutions.
  - Finally, we apply the classical Yamada-Watanabe theorem to ensure the existence of a unique probabilistic strong solution.

[1] D. Breit, Existence theory for stochastic power law fluids, *J. Math. Fluid Mech.*, **17** (2015), 295–326.

The model

We consider the following generalized stochastic Navier-Stokes-Voigt equations driven by a multiplicative Gaussian noise:

$$\begin{cases} \mathrm{d}(\boldsymbol{u}-\kappa\Delta\boldsymbol{u}) = \left[\boldsymbol{f}+\mathrm{div}\big(-\pi\mathbf{I}+\nu|\mathbf{D}(\boldsymbol{u})|^{p-2}\mathbf{D}(\boldsymbol{u})-\boldsymbol{u}\otimes\boldsymbol{u}\big)\right]\mathrm{d}t \\ \qquad\qquad + \Phi(\boldsymbol{u})\mathrm{d}\mathrm{W}(t), \ \text{ in } \mathcal{O}_T, \\ \mathrm{div}\,\boldsymbol{u} = 0, \ \text{ in } \mathcal{O}_T, \\ \boldsymbol{u} = \boldsymbol{u}_0, \ \text{ in } \mathcal{O}\times\{0\}, \\ \boldsymbol{u} = \mathbf{0}, \ \text{ on } \Gamma_T, \end{cases} \tag{GSNSVE}$$

where $\mathcal{O}_T := \mathcal{O}\times(0,T)$ and $\Gamma_T := \partial\mathcal{O}\times[0,T]$,

- $\boldsymbol{u} = (u_1,\ldots,u_d)$ represents the velocity field;
- $\boldsymbol{f} = (f_1,\ldots,f_d)$ is an external vector field;
- π denotes the pressure;
- ν and κ are given positive constants that account for the kinematic viscosity and relaxation time, respectively;
- $\mathbf{D}(\boldsymbol{u}) := \frac{1}{2}\big(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^\top\big)$ denotes the symmetric part of velocity gradient;
- $\mathrm{W}(\cdot)$ is a cylindrical Wiener process.

Physical significance of the model

- The relaxation time κ means that the time required for a viscoelastic fluid to relax from a deformed state back to its equilibrium configuration.
- The power-law index p is a constant that characterizes the flow is assumed to be such that $p \in (1, \infty)$.
- The characterization of flow depends on the value of p in the following manner:
  - For $p \in (1, 2)$, the model describes the shear-thinning fluids, that is, viscosity decreases with increased stress. For example: Nail polish, ketchup, latex paint, etc.
  - For $p = 2$, we obtain the model that governs Newtonian fluids, that is, the viscous stresses arising from its flow are at every point linearly correlated to the local strain rate-the rate of change of its deformation over time. Stresses are proportional to the rate of change of the fluid's velocity. For example: Water, air, alcohol, glycerol, etc.
  - For $p \in (2, \infty)$, the model describes the shear-thickening fluids, that is, viscosity increases with increased stress. For example: Suspension of corn starch in water, candy compounds, etc.

Function spaces

- Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d$, for $2 \leq d \leq 4$, with smooth boundary. Let $\mathrm{C}_0^\infty(\mathcal{O})^d$ denote the space of all infinitely differentiable $\mathbb{R}^d-$valued functions with compact support in $\mathcal{O}$.
- For $p \in [1, \infty)$, we denote by $\mathrm{L}^p(\mathcal{O})^d$, the Lebesgue space consisting of all $\mathbb{R}^d-$valued measurable (equivalence classes of) functions that are $p-$summable over $\mathcal{O}$. The corresponding Sobolev spaces are represented by $\mathrm{W}^{k,p}(\mathcal{O})^d$, for $k \in \mathbb{N}$,
- For $p = 2$, $\mathrm{W}^{k,2}(\mathcal{O})^d$ are Hilbert spaces that we denote by $\mathrm{H}^k(\mathcal{O})^d$.
- We define for $p \in [1, \infty)$ and $k \in \mathbb{N}$,

$$\begin{aligned}
\boldsymbol{\mathcal{V}} &:= \left\{\boldsymbol{u} \in \mathrm{C}_0^\infty(\mathcal{O})^d : \nabla \cdot \boldsymbol{u} = 0\right\}, \\
\mathbf{H} &:= \text{ the closure of } \boldsymbol{\mathcal{V}} \text{ in the Lebesgue space } \mathrm{L}^2(\mathcal{O})^d, \\
\mathrm{L}^p_\sigma(\mathcal{O})^d &:= \text{ the closure of } \boldsymbol{\mathcal{V}} \text{ in the Lebesgue space } \mathrm{L}^p(\mathcal{O})^d, \\
\mathrm{W}^{p,k}_\sigma(\mathcal{O})^d &:= \text{ the closure of } \boldsymbol{\mathcal{V}} \text{ in the Sobolev space } \mathrm{W}^{k,p}(\mathcal{O})^d.
\end{aligned}$$

- For $p = 2$, we denote the space $\mathrm{W}^{p,k}_\sigma(\mathcal{O})^d$ by $\mathbf{V}^k$, and if $k = 1$, we denote it by $\mathbf{V}_p$. If both $p = 2$ and $k = 1$, we denote $\mathrm{W}^{p,k}_\sigma(\mathcal{O})^d$ solely by $\mathbf{V}$.

Function spaces

- Let $(\cdot,\cdot)$ stand for the inner product of the Hilbert space $\mathrm{L}^2(\mathcal{O})^d$, and we denote by $\langle\cdot,\cdot\rangle$, the induced duality product between the space $\mathrm{W}_0^{1,p}(\mathcal{O})^d$ and its dual $\mathrm{W}^{-1,p'}(\mathcal{O})^d$, as well as between $\mathrm{L}^p(\mathcal{O})^d$ and its dual $\mathrm{L}^{p'}(\mathcal{O})^d$, where $\frac{1}{p}+\frac{1}{p'}=1$.
- The L^p, $\mathrm{W}^{k,p}$ and $\mathrm{W}^{-k,p'}$ norms will be denoted in short by $\|\cdot\|_p$, $\|\cdot\|_{k,p}$ and $\|\cdot\|_{-k,p'}$, respectively. By an application of the Poincaré inequality, on $\mathbf{V}$, we consider the norm $\|\boldsymbol{u}\|_{\mathbf{V}}:=\|\nabla\boldsymbol{u}\|_2$, $\boldsymbol{u}\in\mathbf{V}$.
- Let $\mathbf{U}$ be a separable Hilbert space with the inner product denoted by $(\cdot,\cdot)_{\mathbf{U}}$ and associated norm by $\|\cdot\|_{\mathbf{U}}$. and $\mathcal{L}_2(\mathbf{U},\mathrm{L}^2(\mathcal{O})^d)$ denote the space of Hilbert-Schmidt operators from $\mathbf{U}$ to $\mathrm{L}^2(\mathcal{O})^d$ and associated norm by $\|\cdot\|_{\mathcal{L}_2}$.
- Moreover, we define an auxiliary space $\mathbf{U}_0\supset\mathbf{U}$ as

$$\mathbf{U}_0:=\left\{\boldsymbol{v}=\sum_{k\in\mathbb{N}}\alpha_k\mathbf{e}_k:\sum_{k\in\mathbb{N}}\frac{\alpha_k^2}{k^2}<\infty\right\}$$

equipped with the norm

$$\|\boldsymbol{v}\|_{\mathbf{U}_0}^2:=\sum_{k\in\mathbb{N}}\frac{\alpha_k^2}{k^2},\quad \boldsymbol{v}=\sum_{k\in\mathbb{N}}\alpha_k\mathbf{e}_k.$$

Cylindrical Wiener process

- $\mathrm{W}(\cdot)$ is called a Q-Wiener process if
  - $\mathrm{W}(0) = 0$, $\mathbb{P}-$a.s.;
  - $\mathrm{W}(\cdot)$ has continuous trajectories;
  - $\mathrm{W}(\cdot)$ has independent increments;
  - $\mathscr{L}(\mathrm{W}(t) - \mathrm{W}(s)) = \mathcal{N}(0, (t-s)\mathrm{Q}),\ t \geq s \geq 0.$
- $\mathrm{W}(\cdot)$ is a Q-Wiener process. Then, the following hold:
  - $\mathrm{W}(\cdot)$ is a Gaussian process on $\mathbf{U}$ and
$$\mathbb{E}[\mathrm{W}(t)] = 0, \quad \mathrm{Cov}[\mathrm{W}(t)] = t\mathrm{Q}, \quad t \geq 0;$$
  - For arbitrary $t \geq 0$, $\mathrm{W}(\cdot)$ has the expansion
$$\mathrm{W}(t) = \sum_{k \in \mathbb{N}} \sqrt{\eta_k}\beta_k(t)\mathbf{e}_k, \tag{1}$$
where $\beta_k(t) = \frac{1}{\eta_k}(\mathrm{W}(t), \mathbf{e}_k)$, $k \in \mathbb{N}$, are real valued Brownian motions mutually independent on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, $\mathrm{Q}\eta_k = \eta_k\mathbf{e}_k$, $k \in \mathbb{N}$ and the series (1) is convergent in $\mathrm{L}^2(\Omega, \mathscr{F}, \mathbb{P}; \mathbf{U})$;
  - Let $\mathrm{W}(\cdot)$ be a cylindrical Wiener process on $\mathbf{U}$, that is, the covariance operator Q is equals to the identity operator. Then, $\mathrm{W}(\cdot)$ has the representation $\mathrm{W}(t) = \sum\limits_{k \in \mathbb{N}} \beta_k(t)\mathbf{e}_k$.

Assumptions on initial data and forcing term

We shall prove that a weak solution typically exists for given Borel measures Λ_0 and $\Lambda_{\boldsymbol{f}}$ that account for initial and forcing laws as follows:

$$\Lambda_0 = \mathbb{P} \circ \boldsymbol{u}_0^{-1}, \quad \text{i.e.,} \ \ \mathbb{P}\big(\boldsymbol{u}_0 \in \mathrm{U}\big) = \Lambda_0(\mathrm{U}), \ \ \text{for all} \ \ \mathrm{U} \in \mathscr{B}(\mathbf{V}), \tag{ID}$$

$$\Lambda_{\boldsymbol{f}} = \mathbb{P} \circ \boldsymbol{f}^{-1}, \quad \text{i.e.,} \ \ \mathbb{P}\big(\boldsymbol{f} \in \mathrm{U}\big) = \Lambda_{\boldsymbol{f}}(\mathrm{U}), \ \ \text{for all} \ \ \mathrm{U} \in \mathscr{B}(\mathbf{L}^2(\mathcal{O}_T)). \tag{FT}$$

It should be noted that even if the initial datum $\boldsymbol{u}_0$ and the forcing term $\boldsymbol{f}$ are given, they might live on different probability spaces, and therefore $\boldsymbol{u}_0$ and $\boldsymbol{u}(0)$ from one hand, and $\boldsymbol{f}_t$ and $\boldsymbol{f}(t)$ on the other, can only coincide in law. On the initial and forcing laws, we assume that for some constant $\gamma = \gamma(p, d)$

$$\int_{\mathbf{V}} \|\boldsymbol{z}\|_{\mathbf{V}}^{\gamma} \mathrm{d}\Lambda_0(\boldsymbol{z}) < \infty, \tag{IDE}$$

$$\int_{\mathbf{L}^2(\mathcal{O}_T)} \|\mathbf{g}\|_{\mathrm{L}^2(\mathcal{O}_T)}^{\gamma} \mathrm{d}\Lambda_{\boldsymbol{f}}(\mathbf{g}) < \infty. \tag{FTE}$$

Assumption on noise coefficient

We suppose that the noise coefficient $\Phi(\boldsymbol{u})$ satisfies linear growth and Lipschitz conditions. We assume that for each $\boldsymbol{w} \in \mathrm{L}^2(\mathcal{O})^d$ there is a mapping

$$\Phi(\boldsymbol{w}) : \mathbf{U} \longrightarrow \mathrm{L}^2(\mathcal{O})^d \text{ such that } \mathbf{e}_k \longmapsto \Phi(\boldsymbol{w})\mathbf{e}_k = \phi_k(\boldsymbol{w}),$$

where $\{\mathbf{e}_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathbf{U}$, such that $\phi_k \in \mathrm{C}(\mathbb{R}^d)$ and the following conditions hold for some constants $K, L > 0$:

$$\sum_{k\in\mathbb{N}} |\phi_k(\boldsymbol{\xi})| \leq K(1 + |\boldsymbol{\xi}|), \quad \text{and} \quad \sum_{k\in\mathbb{N}} |\phi_k(\boldsymbol{\xi}) - \phi_k(\boldsymbol{\zeta})| \leq L|\boldsymbol{\xi} - \boldsymbol{\zeta}|, \quad \boldsymbol{\xi}, \boldsymbol{\zeta} \in \mathbb{R}^d. \tag{LLC}$$

Moreover, we are assuming that the following condition holds for some constant $C > 0$,

$$\sup_{k\in\mathbb{N}} k^2 |\phi_k(\boldsymbol{\xi})|^2 \leq C(1 + |\boldsymbol{\xi}|^2), \qquad \boldsymbol{\xi} \in \mathbb{R}^d. \tag{2}$$

Probabilistically weak solution

Definition

Let Λ_0 and $\Lambda_{\boldsymbol{f}}$ be Borel probability measures on $\mathbf{V}$ and $\mathbf{L}^2(\mathcal{O}_T)$, respectively. We say that

$$\big((\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P}), \boldsymbol{u}, \boldsymbol{u}_0, \boldsymbol{f}, \mathrm{W}\big)$$

is a probabilistic weak solution to the stochastic problem (GSNSVE), with initial datum Λ_0 and forcing term $\Lambda_{\boldsymbol{f}}$, if:

- $(\Omega, \mathscr{F}, \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration $\{\mathscr{F}_t\}_{t\in[0,T]}$;
- W is a cylindrical $\{\mathscr{F}_t\}_{t\in[0,T]}-$adapted Wiener process;
- $\boldsymbol{u}$ is a progressively $\{\mathscr{F}_t\}_{t\in[0,T]}-$measurable stochastic process with paths $t \mapsto \boldsymbol{u}(t,\omega) \in \mathrm{L}^\infty(0,T;\mathbf{V}) \cap \mathrm{L}^p(0,T;\mathrm{W}_0^{1,p}(\mathcal{O})^d)$, $\mathbb{P}-$a.s. with a continuous modification having $\mathbb{P}-$a.s. paths in $\mathrm{C}([0,T];\mathbf{V})$;
- $\boldsymbol{u}(0)$ $(:= \boldsymbol{u}_0)$ is progressively $\{\mathscr{F}_t\}_{t\in[0,T]}-$measurable on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with $\mathbb{P}-$a.s. paths $\boldsymbol{u}(0,\omega) \in \mathbf{V}$ and $\Lambda_0 = \mathbb{P} \circ \boldsymbol{u}_0^{-1}$ in the sense of (ID);

Probabilistically weak solution

- $\boldsymbol{f}$ is an $\{\mathscr{F}_t\}_{t\in[0,T]}-$adapted stochastic process $\mathbb{P}-$a.s. paths $\boldsymbol{f}(t,\omega)\in\mathbf{L}^2(\mathcal{O}_T)$ and $\Lambda_{\boldsymbol{f}}=\mathbb{P}\circ\boldsymbol{f}^{-1}$ in the sense of (FT);
- for every $\boldsymbol{\varphi}\in\mathrm{C}_0^\infty(\mathcal{O})^d$ with $\operatorname{div}\boldsymbol{\varphi}=0$ and all $t\in[0,T]$, the following identity holds $\mathbb{P}-$a.s.:

$$\begin{aligned}
&\int_{\mathcal{O}}\boldsymbol{u}(t)\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}+\kappa\int_{\mathcal{O}}\nabla\boldsymbol{u}(t):\nabla\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}-\int_0^t\int_{\mathcal{O}}\boldsymbol{u}\otimes\boldsymbol{u}:\nabla\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\mathrm{d}s\\
&\quad+\nu\int_0^t\int_{\mathcal{O}}|\mathbf{D}(\boldsymbol{u})|^{p-2}\mathbf{D}(\boldsymbol{u}):\mathbf{D}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\mathrm{d}s\\
&=\int_{\mathcal{O}}\boldsymbol{u}_0\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}+\kappa\int_{\mathcal{O}}\nabla\boldsymbol{u}_0:\nabla\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\\
&\quad+\int_0^t\int_{\mathcal{O}}\boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\mathrm{d}s+\int_0^t\int_{\mathcal{O}}\Phi(\boldsymbol{u})\mathrm{dW}(s)\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}.
\end{aligned}\tag{3}$$

Probabilistic strong solution

Definition (Probabilistically strong solution)

We are given a stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P})$, initial datum $\boldsymbol{u}_0$ and a forcing term $\boldsymbol{f}$. Then, the problem (GSNSVE) has a pathwise probabilistic strong solution if and only if there exists a $\boldsymbol{u} : [0,T] \times \Omega \to \mathbf{V}$ with paths

$$\boldsymbol{u}(\cdot,\omega) \in \mathrm{L}^\infty(0,T;\mathbf{V}) \cap \mathrm{L}^p(0,T;\mathrm{W}_0^{1,p}(\mathcal{O})^d),\ \mathbb{P}-\text{a.s.},$$

with a continuous modification having $\mathbb{P}-$a.s. paths in $\mathrm{C}([0,T];\mathbf{V})$, and (3) holds for all $\boldsymbol{\phi} \in \mathbf{V}$.

Definition (Pathwise uniqueness)

For $i = 1, 2$, let $\boldsymbol{u}_i$ be any two solutions on the stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P})$ to the system (GSNSVE) with initial datum $\boldsymbol{u}_0$ and forcing term $\boldsymbol{f}$. Then, the solutions of the system (GSNSVE) are pathwise unique if and only if

$$\mathbb{P}\Big\{\boldsymbol{u}_1(t) = \boldsymbol{u}_2(t), \ \ \text{for all} \ \ t \geq 0\Big\} = 1.$$

Existence of a probabilistic weak solution

Theorem (Main theorem-I)

Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary $\partial\mathcal{O}$ of class C^2, and assume that conditions (IDE) *and* (FTE) *hold for*

$$\gamma \geq \max\left\{\frac{pd}{d-2}, 2+\frac{2p}{d-2}\right\} \ (d \neq 2) \quad \text{and} \quad \gamma \geq 2 \ (d=2),$$

and (LLC), (2) *are fulfilled. If $2 \leq d \leq 4$ and*

$$p > \frac{2d}{d+2},$$

then there exists, at least, a probabilistic weak solution

$$\left((\overline{\Omega}, \overline{\mathscr{F}}, \{\overline{\mathscr{F}}_t\}_{t\in[0,T]}, \overline{\mathbb{P}}), \overline{\boldsymbol{u}}, \overline{\boldsymbol{u}}_0, \overline{\boldsymbol{f}}, \overline{\mathrm{W}}\right)$$

in the sense of Definition 1 to the stochastic problem (GSNSVE).

Existence of a unique probabilistic strong solution

Theorem (Main theorem-II)

Under the assumptions of Main theorem-1, there exists a unique probabilistically strong solution to the system (GSNSVE) *in the sense of Definition 2.*

Part-I: Pressure decomposition

- This part is dedicated to the pressure term appearing in the system (GSNSVE).
- We decompose the pressure term in such a way that each part of pressure term corresponds to one term in the equation.
- The idea of such a decomposition has been borrowed from the work[2], where the author extended the results of the work[3] to the stochastic power-law fluids.

[2] D. Breit, Existence theory for stochastic power law fluids, *J. Math. Fluid Mech.*, **17** (2015), 295–326.

[3] J. Wolf, Existence of weak solutions to the equations of nonstationary motion of non- Newtonian fluids with shear-dependent viscosity, *J. Math. Fluid Mech.*, **9** (2007), 104–138.

Pressure decomposition result

Theorem

Let us consider a stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P})$, $\boldsymbol{u} \in \mathrm{L}^2(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^\infty(0,T;\mathbf{V}))$, $\boldsymbol{\mathcal{H}} \in \mathbf{L}^r(\mathcal{O}_T)$ *for some* $1 < r \leq 2$, *both adapted to* $\{\mathscr{F}_t\}_{t\in[0,T]}$. *Moreover, if the initial data* $\boldsymbol{u}_0 \in \mathrm{L}^2(\Omega, \mathscr{F}, \mathbb{P}; \mathbf{V})$ *and* $\Phi \in \mathrm{L}^2(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^\infty(0,T;\mathcal{L}_2(\mathbf{U}, \mathrm{L}^2(\mathcal{O})^d)))$ *progressively measurable such that*

$$\int_{\mathcal{O}} \boldsymbol{u}(t)\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} + \kappa\int_{\mathcal{O}} \nabla\boldsymbol{u}(t) : \nabla\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} + \int_0^t\int_{\mathcal{O}} \boldsymbol{\mathcal{H}} : \nabla\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\mathrm{d}s$$
$$= \int_{\mathcal{O}} \boldsymbol{u}_0\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} + \kappa\int_{\mathcal{O}} \nabla\boldsymbol{u}_0 : \nabla\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} + \int_0^t\int_{\mathcal{O}} \Phi(\boldsymbol{u})\mathrm{d}\mathrm{W}(s)\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x},$$

for all $\boldsymbol{\phi} \in \boldsymbol{\mathcal{V}}$ *and all* $t \in [0,T]$. *Then there are functions* $\pi_{\boldsymbol{\mathcal{H}}}$, π_Φ *and* π_h *adapted to* $\{\mathscr{F}_t\}_{t\in[0,T]}$ *such that the following hold:*

- *We have* $\Delta\pi_h = 0$ *and there holds for* $m := \min\{2, r\}$

$$\mathbb{E}\left[\int_0^T \|\pi_{\boldsymbol{\mathcal{H}}}(t)\|_r^r \mathrm{d}t\right] \leq C\mathbb{E}\left[\int_0^T \|\boldsymbol{\mathcal{H}}(t)\|_r^r \mathrm{d}t\right],$$
$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\pi_\Phi(t)\|_2^2\right] \leq C\mathbb{E}\left[\sup_{t\in[0,T]} \|\Phi(\boldsymbol{u}(t))\|_{\mathcal{L}_2}^2\right],$$

Pressure decomposition result

Theorem

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\pi_h(t)\|_m^m\right]\leq C\mathbb{E}\Bigg[1+\sup_{t\in[0,T]}\left\{\|\boldsymbol{u}(t)\|_2^2+\kappa\|\nabla\boldsymbol{u}(t)\|_2^2\right\}$$

$$+\|\boldsymbol{u}_0\|_2^2+\kappa\|\nabla\boldsymbol{u}_0\|_2^2+\sup_{t\in[0,T]}\|\Phi(\boldsymbol{u}(t))\|_{\mathcal{L}_2}^2+\int_0^T\|\boldsymbol{\mathcal{H}}(t)\|_r^r\mathrm{d}t\Bigg].$$

2. *There holds*

$$\int_{\mathcal{O}}\left(\boldsymbol{u}(t)-\nabla\pi_h(t)\right)\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}+\kappa\int_{\mathcal{O}}\nabla\boldsymbol{u}(t):\nabla\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}+\int_0^t\int_{\mathcal{O}}\boldsymbol{\mathcal{H}}:\nabla\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\mathrm{d}s$$

$$-\int_0^t\int_{\mathcal{O}}\pi_{\boldsymbol{\mathcal{H}}}\operatorname{div}\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\mathrm{d}s$$

$$=\int_{\mathcal{O}}\boldsymbol{u}_0\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}+\kappa\int_{\mathcal{O}}\nabla\boldsymbol{u}_0:\nabla\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}+\int_{\mathcal{O}}\pi_{\Phi}(t)\operatorname{div}\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}$$

$$+\int_0^t\int_{\mathcal{O}}\Phi(\boldsymbol{u})\mathrm{dW}(s)\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x},$$

for all $\boldsymbol{\phi}\in\mathrm{C}_0^\infty(\mathcal{O})^d$. *Moreover* $\pi_h(0)=\pi_{\boldsymbol{\mathcal{H}}}(0)=\pi_\Phi(0)=0,\ \mathbb{P}-$*a.s.*

Term corresponding to π_Φ

Corollary

Assume the conditions of previous Theorem are satisfied. Then there exists $\Phi_\pi \in \mathrm{L}^2(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^\infty(0,T; \mathcal{L}_2(\mathbf{U}, \mathrm{L}^2(\mathcal{O}))))$ *progressively measurable such that*

$$\int_{\mathcal{O}} \pi_\Phi(t)\,\mathrm{div}\,\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} = \int_0^t \int_{\mathcal{O}} \Phi_\pi \mathrm{dW}(s)\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}, \quad \textit{for all } \ \boldsymbol{\phi} \in \mathrm{C}_0^\infty(\mathcal{O}).$$

and $\|\Phi_\pi \mathbf{e}_j\|_2 \leq C(\mathcal{O})\|\Phi\mathbf{e}_j\|_2$*, for all* j*, that is, we have* $\mathbb{P}\otimes\lambda$*, a.e.*

$$\|\Phi_\pi\|_{\mathcal{L}_2} \leq C(\mathcal{O})\|\Phi\|_{\mathcal{L}_2}.$$

Furthermore, if Φ *satisfies* (LLC)*, then there holds for all* $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathrm{L}^2(\mathcal{O})^d$

$$\|\Phi_\pi(\boldsymbol{u}_1) - \Phi_\pi(\boldsymbol{u}_2)\|_{\mathcal{L}_2} \leq C(L,\mathcal{O})\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_2.$$

Corollary

Let the conditions of previous Theorem be satisfied. Then, for all $\gamma \in [1,\infty)$

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\pi_h(t)\|_m^m\right]^\gamma \leq C < \infty.$$

Terms corresponding to $\pi_{\boldsymbol{\mathcal{H}}}$

Corollary

Let the conditions of previous Theorem be satisfied, and assume that the following decomposition holds:

$$\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}_1 + \boldsymbol{\mathcal{H}}_2,$$

where $\boldsymbol{\mathcal{H}}_1 \in \mathrm{L}^{r_1}(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^{r_1}(0,T; \mathrm{L}^{r_1}(\mathcal{O})^{d\times d}))$, $\boldsymbol{\mathcal{H}}_2 \in \mathrm{L}^{r_2}(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^{r_2}(0,T; \mathrm{L}^{r_2}(\mathcal{O})^{d\times d}))$ *and* $\operatorname{div} \boldsymbol{\mathcal{H}}_2 \in \mathrm{L}^{r_2}(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^{r_2}(0,T; \mathrm{L}^{r_2}(\mathcal{O})^{d}))$. *Then, we have*

$$\pi_{\boldsymbol{\mathcal{H}}} = \pi_1 + \pi_2,$$

and there holds for all $\gamma \in [1, \infty)$,

$$\mathbb{E}\left[\int_0^T \|\pi_1(t)\|_{r_1}^{r_1}\mathrm{d}t\right]^{\gamma} \leq C\mathbb{E}\left[\int_0^T \|\boldsymbol{\mathcal{H}}_1(t)\|_{r_1}^{r_1}\mathrm{d}t\right]^{\gamma},$$

$$\mathbb{E}\left[\int_0^T \|\pi_2(t)\|_{r_2}^{r_2}\mathrm{d}t\right]^{\gamma} \leq C\mathbb{E}\left[\int_0^T \|\boldsymbol{\mathcal{H}}_2(t)\|_{r_2}^{r_2}\mathrm{d}t\right]^{\gamma},$$

$$\mathbb{E}\left[\int_0^T \|\nabla\pi_2(t)\|_{r_2}^{r_2}\mathrm{d}t\right]^{\gamma} \leq C\mathbb{E}\left[\int_0^T \left\{\|\boldsymbol{\mathcal{H}}_2(t)\|_{r_2}^{r_2} + \|\operatorname{div} \boldsymbol{\mathcal{H}}_2(t)\|_{r_2}^{r_2}\right\}\mathrm{d}t\right]^{\gamma}.$$

Part-2: Existence of solutions to the auxiliary problem (AP1)

- First, we regularize our problem (GSNSVE), with a stabilization term and call the resultant as auxiliary problem (AP1).
- In order to show the existence of the solutions, we first consider a finite-dimensional approximation and prove that the local solution exists for the finite-dimensional system.
- Then, we establish a uniform energy estimate followed by the existence result where we use compactness arguments, Prokhorov's theorem and Skorokhod's representation theorem.

Auxiliary problem

Let us regularize the problem (GSNSVE), with the following stabilization term in the momentum equation:

$$\alpha\mathbf{a}(\boldsymbol{u}), \qquad \mathbf{a}(\boldsymbol{u}) := |\boldsymbol{u}|^{q-2}\boldsymbol{u}, \qquad \alpha > 0, \qquad 1 < q < \infty. \tag{4}$$

Given $\alpha > 0$, we consider the problem

$$\begin{cases} \mathrm{d}(\mathrm{I} - \kappa\Delta)\boldsymbol{u} = \left\{ \operatorname{div} \mathbf{A}(\boldsymbol{u}) - \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla\pi - \alpha\mathbf{a}(\boldsymbol{u}) + \boldsymbol{f} \right\}\mathrm{d}t + \Phi(\boldsymbol{u})\mathrm{dW}(t), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \end{cases} \tag{AP1}$$

depending on the initial Λ_0 and forcing $\Lambda_{\boldsymbol{f}}$, laws in the conditions of (IDE) and (FTE), respectively and $\mathbf{A}(\boldsymbol{u}) := |\mathbf{D}(\boldsymbol{u})|^{p-2}\mathbf{D}(\boldsymbol{u})$, for $p \in (1, \infty)$. The exponent q in (4) is chosen in such a way that the convective term becomes a compact perturbation. For that purpose, we choose

$$q \geq \max\{2p', 3\}, \tag{Cq}$$

and thus a solution $\boldsymbol{u}$ is expected in the following space:

$$\begin{aligned} \boldsymbol{\mathcal{V}}_{p,q} := \mathrm{L}^2(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^\infty(0, T; \mathbf{V})) \cap \mathrm{L}^p(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^p(0, T; \mathrm{W}_0^{1,p}(\mathcal{O})^d) \\ \cap \mathrm{L}^q(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^q(0, T; \mathrm{L}^q(\mathcal{O})^d)). \end{aligned}$$

Finite-dimensional system

By means of separability, there exists a basis $\{\boldsymbol{\psi}_k\}_{k\in\mathbb{N}}$ of $\mathbf{V}$, formed by the eigenfunctions of a suitable spectral problem, that is, orthogonal in $\mathrm{L}^2(\mathcal{O})^d$ and that can be orthonormal in $\mathrm{W}_0^{1,2}(\mathcal{O})^d$. Given $n\in\mathbb{N}$, let us consider the $n-$dimensional space $\mathbf{X}^n=\mathrm{span}\{\boldsymbol{\psi}_1,\ldots,\boldsymbol{\psi}_n\}$. For each $n\in\mathbb{N}$, we search for approximate solutions of the form

$$\boldsymbol{u}_n(x,t)=\sum_{k=1}^{n}c_k^n(t)\boldsymbol{\psi}_k(x),\quad \boldsymbol{\psi}_k\in\mathbf{X}^n,$$

where the coefficients $c_1^n(t),\ldots,c_n^n(t)$ are solutions of the following n stochastic ordinary differential equations:

$$\begin{aligned}
&\mathrm{d}\Big[\big(\boldsymbol{u}_n(t),\boldsymbol{\psi}_k\big)+\kappa\big(\nabla\boldsymbol{u}_n(t),\nabla\boldsymbol{\psi}_k\big)\Big]\\
&=\Big[\big(\boldsymbol{u}_n(t)\otimes\boldsymbol{u}_n(t):\nabla\boldsymbol{\psi}_k\big)-\nu\big\langle|\mathbf{D}(\boldsymbol{u}_n(t))|^{p-2}\mathbf{D}(\boldsymbol{u}_n(t)):\mathbf{D}(\boldsymbol{\psi}_k)\big\rangle\\
&\quad+\big(\boldsymbol{f}(t),\boldsymbol{\psi}_k\big)\Big]\mathrm{d}t-\alpha\big(\mathbf{a}(\boldsymbol{u}_n(t)),\boldsymbol{\psi}_k\big)+\Phi(\boldsymbol{u}_n(t))\mathrm{d}\mathrm{W}_n(t),\boldsymbol{\psi}_k\big),
\end{aligned}\tag{5}$$

for $k=1,\ldots,n$, supplemented with the initial conditions $\boldsymbol{u}_n(0)=\boldsymbol{u}_0^n$, in $\mathcal{O}$, where $\boldsymbol{u}_0^n=P^n(\boldsymbol{u}_0)$, with P^n denoting the orthogonal projection $P^n:\mathbf{V}\longrightarrow\mathbf{X}^n$ so that

$$\boldsymbol{u}_n(0,x)=\sum_{k=1}^{n}c_k^n(0)\boldsymbol{\psi}_k(x),\quad c_k^n(0)=c_{k,0}^n:=(\boldsymbol{u}_0,\boldsymbol{\psi}_k),\quad k=1,\ldots,n.$$

Uniform estimates-I

Using the monotonicity property[4], we are able to prove the existence of unique strong solution to the finite-dimensional stochastic system (5) upto some local time. In order to prove that the solution is global, we show the following uniform estimate:

Theorem (Uniform estimates-I)

Let $p \in (1, \infty)$, and assume that (LLC) *and* (Cq) *are verified. Assume, in addition, that* (IDE) *and* (FTE) *hold with $\gamma = 2$. Then there exists a positive constant C, neither depending on n nor on α such that*

$$\mathbb{E}\left[\sup_{t\in(0,T)}\left\{\|\boldsymbol{u}_n(t)\|_2^2 + 2\kappa\|\nabla\boldsymbol{u}_n(t)\|_2^2\right\} + 4C(p,\mathcal{O})\nu\int_0^T\|\nabla\boldsymbol{u}_n(t)\|_p^p\mathrm{d}t + 2\alpha\int_0^T\|\boldsymbol{u}_n(t)\|_q^q\mathrm{d}t\right]$$
$$\leq C(\kappa)\left\{\left(\frac{1}{\lambda_1}+\kappa\right)\int_{\mathbf{V}}\|\boldsymbol{z}\|_{\mathbf{V}}^2\mathrm{d}\Lambda_0(\boldsymbol{z}) + \int_{\mathbf{L}^2(\mathcal{O}_T)}\|\mathbf{g}\|_{\mathbf{L}^2(\mathcal{O}_T)}^2\mathrm{d}\Lambda_{\boldsymbol{f}}(\mathbf{g}) + C(K)T\right\}e^{\frac{C(K)T}{\lambda_1}},$$

where λ_1 is the first eigenvalue of the Dirichlet Laplacian.

---

[4] C. Prévôt and M. Röckner, *A Concise Course on Stochastic Partial Differential Equations*, Springer, Berlin Heidelberg, 2007.

Existence of solutions to the auxiliary problem (AP1)

Theorem (Existence of solutions to (AP1))

Let $p \in (1, \infty)$, *and assume the conditions* (LLC) *and* (Cq) *are satisfied. Assume, in addition, that* (IDE) *and* (FTE) *hold with* $\gamma = 2$. *Then there exists a probabilistic weak solution to the approximate system* (AP1), $((\overline{\Omega}, \overline{\mathscr{F}}, \{\overline{\mathscr{F}}_t\}_{t\in[0,T]}, \overline{\mathbb{P}}), \overline{\boldsymbol{u}}, \overline{\boldsymbol{u}}_0, \overline{\boldsymbol{f}}, \overline{\mathrm{W}})$, *defined analogous to* $\big((\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P}), \boldsymbol{u}, \boldsymbol{u}_0, \boldsymbol{f}, \mathrm{W}\big)$ *in the Definition 1.*

Outlines of the proof: Using the Banach-Alaoglu theorem, we obtain the following convergences along some subsequences:

$$
\begin{aligned}
\boldsymbol{u}_n &\xrightarrow{w} \boldsymbol{u}, &&\text{in } \mathrm{L}^2\big(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^\infty(0,T;\mathbf{V})\big),\\
\boldsymbol{u}_n &\xrightarrow{w} \boldsymbol{u}, &&\text{in } \mathrm{L}^p\big(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^p(0,T;\mathrm{W}_0^{1,p}(\mathcal{O})^d\big),\\
\boldsymbol{u}_n &\xrightarrow{w} \boldsymbol{u}, &&\text{in } \mathrm{L}^q\big(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^q(0,T;\mathrm{L}^q(\mathcal{O})^d)\big),\\
\mathbf{a}(\boldsymbol{u}_n) &\xrightarrow{w} \mathbf{a}, &&\text{in } \mathrm{L}^{q'}\big(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^{q'}(0,T;\mathrm{L}^{q'}(\mathcal{O})^d)\big),\\
\boldsymbol{u}_n \otimes \boldsymbol{u}_n &\xrightarrow{w} \boldsymbol{w}, &&\text{in } \mathrm{L}^{\frac{q}{2}}\big(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^{\frac{q}{2}}(0,T;\mathrm{L}^{\frac{q}{2}}(\mathcal{O})^{d\times d})\big),\\
\mathbf{A}(\boldsymbol{u}_n) &\xrightarrow{w} \mathbf{S}, &&\text{in } \mathrm{L}^{p'}\big(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^{p'}(0,T;\mathrm{L}^{p'}(\mathcal{O})^{d\times d})\big),\\
\Phi(\boldsymbol{u}_n) &\xrightarrow{w} \Psi, &&\text{in } \mathrm{L}^2\big(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^2(0,T;\mathcal{L}_2(\mathbf{U},\mathrm{L}^2(\mathcal{O})^d))\big).
\end{aligned}
$$

Identification of limit functions

- Our aim is to establish that

$$\boldsymbol{w} = \boldsymbol{u} \otimes \boldsymbol{u}, \quad \mathbf{S} = \mathrm{A}(\boldsymbol{u}), \quad \text{and} \quad \Psi = \Phi(\boldsymbol{u}).$$

- Now, we test (5) with $\boldsymbol{\phi} \in \boldsymbol{\mathcal{V}}$, so that $\mathbb{P}-$a.s.,

$$\begin{aligned}
&\int_{\mathcal{O}} \boldsymbol{u}_n(t) \cdot \boldsymbol{\phi} \,\mathrm{d}\boldsymbol{x} + \kappa \int_{\mathcal{O}} \nabla \boldsymbol{u}_n(t) : \nabla \boldsymbol{\phi} \,\mathrm{d}\boldsymbol{x} \\
&\equiv \int_{\mathcal{O}} \boldsymbol{u}_n(t) \cdot P_s^n(\boldsymbol{\phi}) \,\mathrm{d}\boldsymbol{x} + \kappa \int_{\mathcal{O}} \nabla \boldsymbol{u}_n(t) : \nabla P_s^n(\boldsymbol{\phi}) \,\mathrm{d}\boldsymbol{x} \\
&= \int_{\mathcal{O}} \boldsymbol{u}_0 \cdot P_s^n(\boldsymbol{\phi}) \,\mathrm{d}\boldsymbol{x} + \kappa \int_{\mathcal{O}} \nabla \boldsymbol{u}_0(t) : \nabla P_s^n(\boldsymbol{\phi}) \,\mathrm{d}\boldsymbol{x} + \int_0^t \int_{\mathcal{O}} \mathbf{G}_n : \nabla P_s^n(\boldsymbol{\phi}) \,\mathrm{d}\boldsymbol{x}\mathrm{d}s \\
&\quad + \int_0^t \int_{\mathcal{O}} \Phi(\boldsymbol{u}_n)\mathrm{d}\mathrm{W}_n(s) \cdot P_s^n(\boldsymbol{\phi}) \,\mathrm{d}\boldsymbol{x}, \qquad (6)
\end{aligned}$$

where P_s^n denotes the projection into the $n-$dimensional space $\mathbf{X}^n$ with respect to the $\mathbf{V}^s$ inner product, and

$$\mathbf{G}_n := \boldsymbol{u}_n \otimes \boldsymbol{u}_n + \nabla\Delta^{-1}\mathbf{a}(\boldsymbol{u}_n) - \nu\mathbf{A}(\boldsymbol{u}_n) + \mathbf{F},$$

with $\mathbf{F}$ chosen in $\mathrm{L}^2(0,T;\mathrm{W}^{1,2}(\mathcal{O})^{d\times d})$ in such a way that $\operatorname{div}\mathbf{F} = -\boldsymbol{f}$ in the weak sense.

Identification of limit functions

First, we show that $\boldsymbol{w} = \boldsymbol{u} \otimes \boldsymbol{u}$ and $\Psi = \Phi(\boldsymbol{u})$.

- By energy estimate, we obtain $\mathbf{G}_n \in \mathrm{L}^{q_0}(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^{q_0}(0,T;\mathrm{L}^{q_0}(\mathcal{O})^{d\times d}))$, for $q_0 := \min\{p', q'\} > 1$, uniformly in n.
- Let us define the functional

$$\mathcal{H}(t, \boldsymbol{\phi}) := \int_0^t \int_{\mathcal{O}} \mathbf{G}_n : \nabla P_s^n(\boldsymbol{\phi}) \,\mathrm{d}\boldsymbol{x}\mathrm{d}s, \qquad \boldsymbol{\phi} \in \boldsymbol{\mathcal{V}}.$$

  As $1 + \frac{2}{q_0} > \frac{1}{q_0} - \frac{1}{2}$ implies the embedding $\mathrm{W}^{\tilde{s},q_0}(\mathcal{O}) \hookrightarrow \mathrm{W}^{s,2}(\mathcal{O})$ for $\tilde{s} \geq s + d\big(1 + \frac{2}{q_0}\big)$, we can use (1) to show that

$$\mathbb{E}\left[\|\mathcal{H}\|_{\mathrm{W}^{1,q_0}(0,T;\mathrm{W}_\sigma^{-\tilde{s},q_0}(\mathcal{O}))}\right] \leq C.$$

- An application of BDG's and Young's inequalities, the energy estimates and the Kolmogorov continuity criterion yield

$$\mathbb{E}\left[\left\|\int_0^t \Phi(\boldsymbol{u}_n(s))\mathrm{d}\mathrm{W}_n(s)\right\|_{\mathrm{C}^\mu([0,T];\mathrm{L}^2(\mathcal{O})^d)}\right] \leq C, \quad \mu := \theta - \frac{1}{q},$$

  for $\frac{1}{q} < \theta < \frac{1}{2}$ if $q > 2$, which is the case due to assumption (Cq).

Tightness arguments

- Using the above information in (6), and still using (IDE) and (FTE), we arrive at

$$\mathbb{E}\left[\|(\mathrm{I}-\kappa\Delta)\boldsymbol{u}_n\|_{\mathrm{C}^{\mu}([0,T];\mathrm{W}_{\sigma}^{-\tilde{s},q_0}(\mathcal{O})^d)}\right] \leq C,$$

  for some positive constant C that does not depend on n.

- For some $\eta > 0$, we further have

$$\mathbb{E}\left[\|(\mathrm{I}-\kappa\Delta)\boldsymbol{u}_n\|_{\mathrm{W}^{\eta,q_0}(0,T;\mathrm{W}_{\sigma}^{-\tilde{s},q_0}(\mathcal{O})^d)}\right] \leq C,$$

  and hence

$$\mathbb{E}\left[\|\boldsymbol{u}_n\|_{\mathrm{W}^{\eta,q_0}(0,T;\mathrm{W}_{\sigma}^{2-\tilde{s},q_0}(\mathcal{O})^d)}\right] \leq C.$$

Tightness arguments

- By a version[5] of the Aubin-Lions compactness lemma, we further have

$$\mathrm{W}^{\eta,q_0}(0,T;\mathrm{W}^{2-\tilde{s},q_0}_{\sigma}(\mathcal{O})^d)\cap \mathrm{L}^{\infty}(0,T;\mathrm{L}^{\rho}_{\sigma}(\mathcal{O})^d)\cap \mathrm{L}^{p}(0,T;\mathrm{W}^{1,p}_{0}(\mathcal{O})^d)$$
$$\hookrightarrow\hookrightarrow \mathrm{L}^{\rho}(0,T;\mathrm{L}^{\rho}_{\sigma}(\mathcal{O})^d), \text{ for } \ q_0\leq \rho< p^*,$$

  where $p^*=\frac{dp}{d-p}$.

- The joint law of $\boldsymbol{u}_n, \mathrm{W}, \boldsymbol{u}_0$ and $\boldsymbol{f}$ denote by ϱ_n is tight in the space

$$\mathfrak{V}:=\mathrm{L}^{\rho}(0,T;\mathrm{L}^{\rho}_{\sigma}(\mathcal{O})^d)\otimes \mathrm{C}([0,T];\mathbf{U}_0)\otimes \mathbf{V}\otimes \mathbf{L}^2(\mathcal{O}_T)$$

  and hence ϱ_n has a weakly convergent subsequence[6] with weak limit ϱ.

[5] F. Flandoli and D. Gatarek, Martingale and stationary solutions for stochastic Navier-Stokes equations, *Probab. Theory Related Fields*, **102** (1995), 367–391.

[6] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd edn., North-Holland Mathematical Library 24, North-Holland, Amsterdam (1989).

New probability space and convergences

By Skorokhod's representation theorem[7], there exists a probability space $(\overline{\Omega}, \overline{\mathscr{F}}, \overline{\mathbb{P}})$, a random sequence $(\overline{\boldsymbol{u}}_n, \overline{\mathrm{W}}_n, \overline{\boldsymbol{u}}_0^n, \overline{\boldsymbol{f}}_n)$ and a random variable $(\overline{\boldsymbol{u}}, \overline{\mathrm{W}}, \overline{\boldsymbol{u}}_0, \overline{\boldsymbol{f}})$ on the probability space $(\overline{\Omega}, \overline{\mathscr{F}}, \overline{\mathbb{P}})$, taking values in $\mathfrak{V}$ such that the following hold:

1. The laws of sequence of random variables $(\overline{\boldsymbol{u}}_n, \overline{\mathrm{W}}_n, \overline{\boldsymbol{u}}_0^n, \overline{\boldsymbol{f}}_n)$ and the random variable $(\overline{\boldsymbol{u}}, \overline{\mathrm{W}}, \overline{\boldsymbol{u}}_0, \overline{\boldsymbol{f}})$ under the new probability measure $\overline{\mathbb{P}}$ coincide with ϱ_n and $\varrho := \lim\limits_{n\to\infty} \varrho_n$, respectively;
2. Applying Vitali's convergence theorem, the following convergence results hold:

$$
\begin{cases}
\overline{\mathrm{W}}_n \to \overline{\mathrm{W}}, & \text{in } \mathrm{L}^2(\overline{\Omega}, \overline{\mathscr{F}}, \overline{\mathbb{P}}; \mathrm{C}([0,T]; \mathbf{U}_0)), \\
\overline{\boldsymbol{u}}_n \to \overline{\boldsymbol{u}}, & \text{in } \mathrm{L}^\rho(\overline{\Omega}, \overline{\mathscr{F}}, \overline{\mathbb{P}}; \mathrm{L}^\rho(0,T; \mathrm{L}^\rho_\sigma(\mathcal{O})^d)), \\
\overline{\boldsymbol{u}}_0^n \to \overline{\boldsymbol{u}}_0, & \text{in } \mathrm{L}^2(\overline{\Omega}, \overline{\mathscr{F}}, \overline{\mathbb{P}}; \mathbf{V}), \\
\overline{\boldsymbol{f}}_n \to \overline{\boldsymbol{f}}, & \text{in } \mathrm{L}^2(\overline{\Omega}, \overline{\mathscr{F}}, \overline{\mathbb{P}}; \mathrm{L}^2(0,T; \mathrm{L}^2(\mathcal{O})^d)),
\end{cases}
$$

for ρ and p satisfying the following conditions:

$$
\min\{p', q'\} < \rho < \min\{2^*, p^*, q\}, \qquad p > \frac{2d}{d+2},
$$

obtained from the compactness arguments.

[7] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd edn., North-Holland Mathematical Library 24, North-Holland, Amsterdam (1989).

Verification of limit functions

1. First, we define a new filtration on the newly constructed probability space.
2. Then, our goal is to show that the approximate equations also hold in the newly constructed probability space.
3. In order to fulfill our goal, we first identify the quadratic variation of the martingales as well as cross variation with the limit Wiener process obtained through compactness.
4. Finally, using a convergence result[8], we are able to pass the limit $n \to \infty$, which leads to the verification of $\boldsymbol{w} = \boldsymbol{u} \otimes \boldsymbol{u}$ and $\Psi = \Phi(\boldsymbol{u})$.
5. *Verification of* $\overline{\mathbf{S}} = \mathbf{A}(\overline{\boldsymbol{u}})$*:* Applying the finite and infinite-dimensional Itô formulae to the processes $\|(\mathrm{I} - \kappa\Delta)^{\frac{1}{2}}\overline{\boldsymbol{u}}_n(\cdot)\|_2^2$ and $\|(\mathrm{I} - \kappa\Delta)^{\frac{1}{2}}\overline{\boldsymbol{u}}(\cdot)\|_2^2$, respectively, and then subtracting the former from the later, and then expectation followed by lower semicontinuity, monotonicity[9] of the operators $\mathbf{a}(\cdot)$ and $\mathbf{A}(\cdot)$ lead to the required result.

[8] A. Debussche, N. Glatt-Holtz and R. Temam, Local Martingale and pathwise solutions for an abstract fluids model, *Phys. D Nonlinear Phenom.*, **240** (2011), 1123–1144.

[9] B. Levant, F. Ramos and E. S. Titi, On the statistical properties of the 3D incompressible Navier-Stokes-Voigt model, *Commun. Math. Sci.*, **8** (2010), 277–293.

Uniform estimate-II

Corollary

Let the hypothesis of Theorem (Existence of solutions to (AP1)*) be verified. In addition, assume that* (IDE) *and* (FTE) *hold with $\gamma \geq 2$. Then there exists a probabilistic weak solution to the system* (AP1) *such that*

$$\mathbb{E}\bigg[\sup_{t\in[0,T]}\left\{\|\overline{\boldsymbol{u}}(t)\|_2^2+\kappa\|\nabla\overline{\boldsymbol{u}}(t)\|_2^2\right\}\bigg]^{\frac{\gamma}{2}}+C(p,\mathcal{O})\nu\mathbb{E}\bigg[\int_0^T\|\nabla\overline{\boldsymbol{u}}(t)\|_p^p\mathrm{d}t\bigg]^{\frac{\gamma}{2}}$$

$$+C\alpha\mathbb{E}\bigg[\int_0^T\|\overline{\boldsymbol{u}}(t)\|_q^q\mathrm{d}t\bigg]^{\frac{\gamma}{2}}$$

$$\leq C_1\Bigg\{\left(\frac{1}{\lambda_1}+\kappa\right)^{\frac{\gamma}{2}}\left(\int_{\mathbf{V}}\|\overline{\boldsymbol{z}}\|_{\mathbf{V}}^2\mathrm{d}\Lambda_0(\overline{\boldsymbol{z}})\right)^{\frac{\gamma}{2}}+\left(\int_{\mathbf{L}^2(\mathcal{O}_T)}\|\overline{\mathbf{g}}\|_{\mathbf{L}^2(\mathcal{O}_T)}^2\mathrm{d}\Lambda_{\overline{\boldsymbol{f}}}(\overline{\mathbf{g}})\right)^{\frac{\gamma}{2}}$$

$$+C(\gamma,K,T)\Bigg\},$$

where the constant C_1 is independent of α.

Part-3: Non-stationary flows

Finally, our goal is to establish the existence of probabilistic weak solutions to the problem (GSNSVE). Our idea will be the same as in the previous part, the only difference will be the reconstruction of pressure term.

1. We start by approximating the original problem by an approximate system under the conditions of previous part.
2. From the existence results of the auxiliary problem (AP1), we obtain the existence of solutions to the approximate system (AS).
3. Later, we establish some useful estimates, followed by weak convergence of subsequences as a direct application of the Banach-Alaoglu theorem.
4. We establish some compactness arguments for the solutions to the approximate system (AS).
5. Finally, we pass the limit in the viscous term with the help of monotone operator theory.

Approximate system

We consider the following system:

$$\begin{cases} \mathrm{d}(\mathrm{I}-\kappa\Delta)\boldsymbol{u}_n(t) = \Big[\mathrm{div}(\mathbf{A}(\boldsymbol{u}_n(t))) - \mathrm{div}(\boldsymbol{u}_n(t)\otimes\boldsymbol{u}_n(t)) - \frac{1}{n}|\boldsymbol{u}_n(t)|^{q-2}\boldsymbol{u}_n(t) \\ \qquad\qquad + \nabla\pi(t) + \boldsymbol{f}_n(t)\Big]\mathrm{d}t + \Phi(t,\boldsymbol{u}_n(t))\mathrm{dW}(t), \\ \boldsymbol{u}_n(0) = \boldsymbol{u}_0^n. \end{cases} \tag{AS}$$

From Part-2 and $\alpha = \frac{1}{n}$, we have the existence of a probabilistic weak solution $\big((\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in[0,T]},\mathbb{P}),\boldsymbol{u}_n,\boldsymbol{u}_0^n,\boldsymbol{f}_n,\mathrm{W}\big)$ to (AS). Therefore, we can write for all $t\in[0,T]$, $\mathbb{P}-$a.s.,

$$\begin{aligned} &\int_{\mathcal{O}}\boldsymbol{u}_n(t)\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} + \kappa\int_{\mathcal{O}}\nabla\boldsymbol{u}_n(t):\nabla\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} + \nu\int_0^t\int_{\mathcal{O}}\mathbf{A}(\boldsymbol{u}_n):\mathbf{D}(\boldsymbol{\phi})\,\mathrm{d}\boldsymbol{x}\mathrm{d}s \\ &\quad + \frac{1}{n}\int_0^t\int_{\mathcal{O}}\mathbf{a}(\boldsymbol{u}_n)\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\mathrm{d}s \\ &= \int_{\mathcal{O}}\boldsymbol{u}_0^n\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} + \kappa\int_{\mathcal{O}}\nabla\boldsymbol{u}_0^n:\nabla\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} + \int_0^t\int_{\mathcal{O}}\boldsymbol{u}_n\otimes\boldsymbol{u}_n:\nabla\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\mathrm{d}s + \int_0^t\int_{\mathcal{O}}\boldsymbol{f}_n\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\mathrm{d}s \\ &\quad + \int_0^t\int_{\mathcal{O}}\Phi(\boldsymbol{u}_n)\mathrm{dW}(s)\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}, \quad \text{for all } \ \boldsymbol{\phi}\in\boldsymbol{\mathcal{V}}. \end{aligned}$$

Again from Part-2, we obtain the uniform estimates for $\boldsymbol{u}_n$ in the space $\mathrm{L}^2(\Omega,\mathscr{F},\mathbb{P};\mathrm{L}^\infty(0,T;\mathbf{V}))\cap\mathrm{L}^p(\Omega,\mathscr{F},\mathbb{P};\mathrm{L}^p(0,T;\mathrm{W}_0^{1,p}(\mathcal{O})^d)$.

Useful estimates

By Uniform estimates-II and assumption (IDE) and (FTE) on Λ_0 and $\Lambda_{\boldsymbol{f}_n}$, with $\gamma \geq 2$, respectively, we find

$$\mathbb{E}\left[\sup_{t\in(0,T)}\left\{\|\boldsymbol{u}_n(t)\|_2^2+\kappa\|\nabla\boldsymbol{u}_n(t)\|_2^2\right\}^{\frac{\gamma}{2}}\right]+C(p,\mathcal{O})\nu\mathbb{E}\left[\int_0^T\|\nabla\boldsymbol{u}_n(t)\|_p^p\mathrm{d}t\right]^{\frac{\gamma}{2}}$$
$$+\frac{C}{n}\mathbb{E}\left[\int_0^T\|\boldsymbol{u}_n(t)\|_q^q\mathrm{d}t\right]^{\frac{\gamma}{2}}\leq C_1(\kappa,\gamma,\lambda_1,K,T).$$

Using the above estimates, we obtain the following:

1. For any $p \geq 1$ and

$$\gamma \geq \max\left\{\frac{pd}{d-2}, 2+\frac{2p}{d-2}\right\}$$

there holds

$$\mathbb{E}\left[\int_0^T\|\boldsymbol{u}_n(t)\|_\rho^\rho\mathrm{d}t\right]\leq C,\ \ \rho:=\frac{pd}{d-2}$$ if $d\neq 2$, and any $\rho\in[1,\infty)$ if $d=2$.

2. For some $\gamma \geq \frac{2d}{d-2}$, there holds

$$\mathbb{E}\left[\int_0^T\|\boldsymbol{u}_n(t)\otimes\boldsymbol{u}_n(t)\|_{q_0}^{q_0}\mathrm{d}t+\int_0^T\|\operatorname{div}(\boldsymbol{u}_n(t)\otimes\boldsymbol{u}_n(t))\|_{q_0}^{q_0}\mathrm{d}t\right]\leq C, \tag{7}$$

for $1\leq q_0\leq \frac{d}{d-1}$.

Weak convergences

Applying the Banach-Alaoglu theorem, we can pass the limit along a subsequence (still denoting by the same index) as follows:

$$
\begin{aligned}
\boldsymbol{u}_n &\xrightarrow{w} \boldsymbol{u}, \quad \text{in} \quad \mathrm{L}^{\frac{\gamma p}{2}}\big(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^p(0,T;\mathrm{W}_0^{1,p}(\mathcal{O})^d)\big), \\
\boldsymbol{u}_n &\xrightarrow{w} \boldsymbol{u}, \quad \text{in} \quad \mathrm{L}^{\gamma}\big(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^\rho(0,T;\mathbf{V})\big), \quad \text{for all} \quad \rho \geq 1, \\
\frac{1}{n}|\boldsymbol{u}_n|^{q-2}\boldsymbol{u}_n &\to 0, \quad \text{in} \quad \mathrm{L}^{\frac{\gamma q'}{2}}(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^{q'}(0,T;\mathrm{L}^{q'}(\mathcal{O})^d)), \\
\boldsymbol{u}_n \otimes \boldsymbol{u}_n &\xrightarrow{w} \boldsymbol{w}, \quad \text{in} \quad \mathrm{L}^{q_0}\big(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^{q_0}(0,T;\mathrm{W}^{1,q_0}(\mathcal{O})^{d\times d}\big), \\
\mathbf{A}(\boldsymbol{u}_n) &\xrightarrow{w} \mathbf{S}, \quad \text{in} \quad \mathrm{L}^{p'}\big(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^{p'}(0,T;\mathrm{L}^{p'}(\mathcal{O})^{d\times d})\big), \\
\mathbf{A}(\boldsymbol{u}_n) &\xrightarrow{w} \mathbf{S}, \quad \text{in} \quad \mathrm{L}^{p'}\big(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^{p'}(0,T;\mathrm{W}^{-1,p'}(\mathcal{O})^{d\times d})\big), \\
\Phi(\boldsymbol{u}_n) &\xrightarrow{w} \widehat{\Phi}, \quad \text{in} \quad \mathrm{L}^{2}\big(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^{\rho}(0,T;\mathcal{L}_2(\mathbf{U},\mathrm{L}^2(\mathcal{O})^d))\big), \quad \text{for all} \quad \rho \geq 1.
\end{aligned} \tag{8}
$$

Moreover, we have

$$
\begin{aligned}
&\boldsymbol{u} \in \mathrm{L}^{\gamma}(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^\infty(0,T;\mathbf{V})), \\
&\widehat{\Phi} \in \mathrm{L}^{\gamma}(\Omega, \mathscr{F}, \mathbb{P}; \mathrm{L}^\infty(0,T;\mathcal{L}_2(\mathbf{U},\mathrm{L}^2(\mathcal{O})^d))).
\end{aligned}
$$

Reconstruction of the pressure term

Define $\boldsymbol{\mathcal{H}}_1^n := \mathbf{A}(\boldsymbol{u}_n)$, $\boldsymbol{\mathcal{H}}_2^n := \boldsymbol{u}_n \otimes \boldsymbol{u}_n + \nabla\Delta^{-1}\boldsymbol{f}_n + \nabla\Delta^{-1}\left(\frac{1}{n}|\boldsymbol{u}_n|^{q-2}\boldsymbol{u}_n\right)$, $\Phi^n := \Phi(\boldsymbol{u}_n)$. From Part-1, we get the functions π_h^n, π_1^n and π_2^n adapted to $\{\overline{\mathscr{F}}_t\}_{t\in[0,T]}$ and Φ_π^n progressively measurable such that $\mathbb{P}-$a.s.,

$$\begin{aligned}
&\int_{\mathcal{O}} (\boldsymbol{u}_n - \nabla\pi_h^n)(t)\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} + \kappa\int_{\mathcal{O}} \nabla\boldsymbol{u}_n(t) : \nabla\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} \\
&= \int_{\mathcal{O}} \boldsymbol{u}_0^n\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} + \kappa\int_{\mathcal{O}} \nabla\boldsymbol{u}_0^n : \nabla\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} - \nu\int_0^t\int_{\mathcal{O}} (\boldsymbol{\mathcal{H}}_1^n - \pi_1^n\mathrm{I}) : \nabla\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\mathrm{d}s \\
&\quad + \int_0^t\int_{\mathcal{O}} \mathrm{div}\left(\boldsymbol{\mathcal{H}}_2^n - \pi_2^n\mathrm{I}\right) : \boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\mathrm{d}s + \int_0^t\int_{\mathcal{O}} \Phi^n\mathrm{dW}(s)\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} + \int_0^t\int_{\mathcal{O}} \Phi_\pi^n\mathrm{dW}(s)\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Hence, the following functions are uniformly bounded, with respect to n, in the below mentioned spaces

$$\begin{aligned}
\boldsymbol{\mathcal{H}}_1^n &\in \mathrm{L}^{\frac{\gamma p'}{2}}(\Omega,\mathscr{F},\mathbb{P};\mathrm{L}^{p'}(0,T;\mathrm{L}^{p'}(\mathcal{O})^{d\times d})), \\
\boldsymbol{\mathcal{H}}_2^n &\in \mathrm{L}^{q_0}(\Omega,\mathscr{F},\mathbb{P};\mathrm{L}^{q_0}(0,T;\mathrm{W}^{1,q_0}(\mathcal{O})^{d\times d})), \\
\Phi^n &\in \mathrm{L}^{\gamma}(\Omega,\mathscr{F},\mathbb{P};\mathrm{L}^{\infty}(0,T;\mathcal{L}_2(\mathbf{U},\mathrm{L}^2(\mathcal{O})^d)),
\end{aligned}$$

where we have used the continuity of $\nabla\Delta^{-1}$ from $\mathrm{L}^{q_0}(\mathcal{O})^d$ to $\mathrm{W}^{1,q_0}(\mathcal{O})^{d\times d}$.

Weak convergences

We know that the corresponding pressure functions are also uniformly bounded in the scalar spaces, that is,

$$
\begin{aligned}
\pi_h^n &\in \mathrm{L}^{\gamma}(\Omega,\mathscr{F},\mathbb{P};\mathrm{L}^{\infty}(0,T;\mathrm{L}^2(\mathcal{O}))),\\
\pi_1^n &\in \mathrm{L}^{\frac{\gamma p'}{2}}(\Omega,\mathscr{F},\mathbb{P};\mathrm{L}^{p'}(0,T;\mathrm{L}^{p'}(\mathcal{O}))),\\
\pi_2^n &\in \mathrm{L}^{q_0}(\Omega,\mathscr{F},\mathbb{P};\mathrm{L}^{q_0}(0,T;\mathrm{W}^{1,q_0}(\mathcal{O}))),\\
\Phi_\pi^n &\in \mathrm{L}^{\gamma}(\Omega,\mathscr{F},\mathbb{P};\mathrm{L}^{\infty}(0,T;\mathcal{L}_2(\mathbf{U},\mathrm{L}^2(\mathcal{O})^d)),
\end{aligned}
$$

where we have used Part-1.Using the regularity theory for the harmonic functions and Corollary 10 to the harmonic pressure term, we find that

$$
\pi_h^n \in \mathrm{L}^{\gamma}(\Omega,\mathscr{F},\mathbb{P};\mathrm{L}^{\rho}(0,T;\mathrm{W}^{k,\infty}(\mathcal{O}))), \quad \text{for all } k\in\mathbb{N}, \quad \text{for all } \rho\geq 1.
$$

Passing $n\to\infty$ along subsequences, we obtain the following convergence results:

$$
\begin{aligned}
\pi_h^n &\xrightarrow{w} \pi_h, &&\text{in} \quad \mathrm{L}^{\gamma}(\Omega,\mathscr{F},\mathbb{P};\mathrm{L}^{\rho}(0,T;\mathrm{W}^{k,\rho}(\mathcal{O})), \quad \text{for all } \rho\geq 1,\\
\pi_1^n &\xrightarrow{w} \pi_1, &&\text{in} \quad \mathrm{L}^{\frac{\gamma p'}{2}}(\Omega,\mathscr{F},\mathbb{P};\mathrm{L}^{p'}(0,T;\mathrm{L}^{p'}(\mathcal{O}))),\\
\pi_2^n &\xrightarrow{w} \pi_2, &&\text{in} \quad \mathrm{L}^{q_0}(\Omega,\mathscr{F},\mathbb{P};\mathrm{L}^{q_0}(0,T;\mathrm{W}^{1,q_0}(\mathcal{O}))),\\
\Phi_\pi^n &\xrightarrow{w} \Phi_\pi, &&\text{in} \quad \mathrm{L}^{\gamma}(\Omega,\mathscr{F},\mathbb{P};\mathrm{L}^{\rho}(0,T;\mathcal{L}_2(\mathbf{U},\mathrm{L}^2(\mathcal{O})^d)), \quad \text{for all } \rho\geq 1.
\end{aligned}
$$

Identification of limit functions

As in Part-2, our aim is to show that $\overline{\mathbf{S}} = \mathbf{A}(\overline{\boldsymbol{u}})$, $\boldsymbol{w} = \boldsymbol{u} \otimes \boldsymbol{u}$, and $\widehat{\Phi} = \Phi(\boldsymbol{u})$. For that purpose, we use compactness arguments and a version of Skorokhod's theorem[10].

Compactness

1. One can note that, now we are dealing with pressure terms also. We use a generalization of this result that includes weak topologies in Banach space: the Jakubowski-Skorokhod theorem will be useful in our case.
2. We establish that the joint law of $\boldsymbol{u}_n, \pi_h^n, \pi_1^n, \pi_2^n, \Phi_\pi^n, \mathrm{W}_n, \boldsymbol{u}_0^n, \boldsymbol{f}_n$ denoted by ϱ_n is tight on the space $\mathfrak{V} :=$ $\mathrm{L}^\rho(0,T;\mathrm{L}^\rho_\sigma(\mathcal{O})^d) \otimes \mathrm{L}^\rho(0,T;\mathrm{L}^\rho(\mathcal{O})) \otimes \mathrm{L}^{p'}_{\mathrm{w}}(0,T;\mathrm{L}^{p'}(\mathcal{O})) \otimes \mathrm{L}^{q_0}_{\mathrm{w}}(0,T;\mathrm{W}^{1,q_0}(\mathcal{O})) \otimes$ $\mathrm{L}^\rho_{\mathrm{w}}(0,T;\mathcal{L}_2(\mathbf{U},\mathrm{L}^2(\mathcal{O})^d)) \otimes \mathrm{C}([0,T];\mathbf{U}_0) \otimes \mathbf{V} \otimes \mathbf{L}^2(\mathcal{O}_T)$.
3. Thus, we can construct a new probability space and a sequence of new random variable which converges $\overline{\mathbb{P}}$-a.s.
4. Then we define the new filtration and validates the approximate system in new probability space using the same procedure as in previous part.
5. Later, we obtain several weak and strong convergences along some subsequences, which we are not mentioning here because of several technicalities.

[10] A. Jakubowski, The almost sure Skorokhod representation for subsequences in nonmetric spaces, *Teor. Veroyatnost. i Primenen*, **42** (1997), 209–216.

Pathwise uniqueness

We rewrite the equation (GSNSVE) as follows:

$$\mathrm{d}(\mathrm{I}-\kappa\Delta)^{\frac{1}{2}}\boldsymbol{u} = (\mathrm{I}-\kappa\Delta)^{-\frac{1}{2}}\Big\{\operatorname{div}\Big(\nu\mathbf{A}(\boldsymbol{u}) - (\boldsymbol{u}\otimes\boldsymbol{u}) - \pi\mathrm{I} - \mathbf{F}\Big)\Big\}\mathrm{d}t + (\mathrm{I}-\kappa\Delta)^{-\frac{1}{2}}\Phi(\boldsymbol{u})\mathrm{d}\mathrm{W}(t),$$

where $\mathbf{F}\in\mathrm{L}^2(0,T;\mathrm{W}^{1,2}(\mathcal{O})^{d\times d})$ such that $\operatorname{div}\mathbf{F} = -\boldsymbol{f}$.

Theorem (Uniqueness)

Under the assumptions of Theorem (Main theorem-I), solution of the system (GSNSVE) *is pathwise unique.*
Moreover, there exists a unique probabilistic strong solution to the system (GSNSVE) *in the sense of Definition 2.*

Outlines of proof:

- Let $\boldsymbol{u}_1(\cdot)$, $\boldsymbol{u}_2(\cdot)$ be any two solutions to the system (GSNSVE), with the initial data $\boldsymbol{u}_0^2$ and $\boldsymbol{u}_0^2$, respectively.
- Define $\boldsymbol{w}(\cdot) := \boldsymbol{u}_1(\cdot) - \boldsymbol{u}_2(\cdot)$ and $\tilde{\Phi}(\cdot) := \Phi(\boldsymbol{u}_1(\cdot)) - \Phi(\boldsymbol{u}_2(\cdot))$.
- Then, $\boldsymbol{w}(\cdot)$ satisfies the following system with the initial data $\boldsymbol{w}(0) = \boldsymbol{u}_0^1 - \boldsymbol{u}_0^2$,

$$\begin{aligned}\mathrm{d}(\mathrm{I}-\kappa\Delta)^{\frac{1}{2}}\boldsymbol{w} = (\mathrm{I}-\kappa\Delta)^{-\frac{1}{2}}\Big[\mathrm{div}\left(\mathbf{A}(\boldsymbol{u}_1) - \mathbf{A}(\boldsymbol{u}_2)\right) - \mathrm{div}\left((\boldsymbol{u}_1 \otimes \boldsymbol{u}_1) - (\boldsymbol{u}_2 \otimes \boldsymbol{u}_2)\right) \\ - \mathrm{div}\left((\pi_1 - \pi_2)\mathrm{I}\right)\Big]\mathrm{d}t + (\mathrm{I}-\kappa\Delta)^{-\frac{1}{2}}\tilde{\Phi}\mathrm{dW},\end{aligned}$$

  a.e. $t \in [0, T]$ in $\mathbf{L}^{p'}$.
- Let us define

$$\varphi(t) := \exp\left(-C_1 \int_0^t \|\nabla \boldsymbol{u}_2(s)\|_2 \mathrm{d}s\right),$$

  where C_1 is some positive constant.
- We conclude the final result by applying infinite-dimensional Itô's formula to the process $\varphi(\cdot)\|(\mathrm{I}-\kappa\Delta)^{\frac{1}{2}}\boldsymbol{w}(\cdot)\|_2^2$.

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Thank you all for your kind attention!