Variation index, Hölder exponent, and Schauder coefficients

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June 2024

Motivation

Pathwise Itô Calculus

Fix a sequence of partitions $\pi = (\pi^n)_{n \geq 0}$ of $[0, T]$:

$$\pi^n = (0 = t_0^n < t_1^n < \cdots < t_{N(\pi^n)}^n = T).$$

Theorem (Föllmer, 1981)

For $f \in C^2(\mathbb{R}, \mathbb{R})$ and $x \in C^0([0, T], \mathbb{R}) \cap Q_\pi^{(2)}([0, T], \mathbb{R})$, the limit of Riemann sums

$$\int_0^t f'(x(s))\, d^\pi x(s) := \lim_{n \to \infty} \sum_{\substack{t_j^n \in \pi^n \\ t_j^n \leq t}} f'(x(t_j^n)) \left(x(t_{j+1}^n) - x(t_j^n) \right)$$

exists and the following identity holds:

$$f(x(t)) = f(x(0)) + \int_0^t f'(x(s))\, d^\pi x(s) + \frac{1}{2} \int_0^t f''(x(s))\, d[x]_\pi^{(2)}(s).$$

p-th variation of continuous functions along π

Definition

$x \in C^0([0,T],\mathbb{R})$ has finite quadratic variation along π, if the sequence of functions

$$[x]^{(2)}_{\pi^n}(t) = \sum_{\substack{t_j^n \in \pi^n \\ t_j^n \le t}} \left| x(t^n_{j+1}) - x(t^n_j) \right|^2 \xrightarrow{n\to\infty} [x]^{(2)}_{\pi}(t)$$

converges uniformly on $[0,T]$ to a continuous (non-decreasing) function $[x]^{(2)}_{\pi}$. In this case, we denote $x \in Q^{(2)}_{\pi}([0,T],\mathbb{R})$.

When a similar condition holds for some $p \in (1,\infty)$, instead of 2, we say x has finite p-th variation along π, or $x \in Q^{(p)}_{\pi}([0,T],\mathbb{R})$.

Recent generalizations of FÖLLMER's pathwise calculus

- CONT & PERKOWSKI (2019):
  higher-order pathwise Itô formula for $p \in 2\mathbb{N}$.

- KIM (2022):
  higher-order pathwise Tanaka-Meyer formula for $p \in 2\mathbb{N}$ with a relevant notion of pathwise local time.

- CONT & JIN (2024):
  fractional pathwise Itô formula for general $p \geq 1$, with a fractional Itô remainder term.

All of these results require the existence of (finite) p-th variation along a given partition sequence π, i.e., we can only develop pathwise calculus for $x \in Q_\pi^{(p)}([0, T], \mathbb{R})$.

The space $Q_\pi^{(p)}([0, T], \mathbb{R})$

However, we *barely* know the space $Q_\pi^{(p)}([0, T], \mathbb{R})$.

The function space $Q_\pi^{(p)}([0, T], \mathbb{R})$ is NOT a vector space.

Proposition (Schied, 2016)

There is an example of continuous functions x and y such that $[x]_{\mathbb{T}}^{(2)}$ and $[y]_{\mathbb{T}}^{(2)}$ exist, but $[x+y]_{\mathbb{T}}^{(2)}$ does not exist.

The construction of this example uses Schauder representation of continuous functions along the dyadic partition sequence $\mathbb{T} = (\mathbb{T}^n)$, where

$$\mathbb{T}^n = \{0 < \frac{1}{2^n} < \frac{2}{2^n} < \dots < \frac{T2^n}{2^n}\}.$$

The space $Q_\pi^{(p)}([0,T],\mathbb{R})$

This indicates that requiring the existence of p-th variation, i.e., the existence of the following limit, would be too strong:

$$\lim_{n\to\infty}[x]_{\pi^n}^{(p)}(t) = \lim_{n\to\infty} \sum_{\substack{t_j^n\in\pi^n \\ t_j^n\le t}} \left|x(t_{j+1}^n) - x(t_j^n)\right|^p = [x]_\pi^{(p)}(t)$$

Thus, we rather consider sup of $[x]_{\pi^n}^{(p)}(T)$, and define

$$\|x\|_\pi^{(p)} := |x(0)| + \sup_{n\ge 0}\left([x]_{\pi^n}^{(p)}(T)\right)^{\frac{1}{p}},$$

$$\mathscr{X}_\pi^{(p)} := \{x \in C^0([0,T],\mathbb{R}) \,:\, \|x\|_\pi^{(p)} < \infty\}.$$

Proposition

For any $p \ge 1$ and a refining partition sequence π with vanishing mesh, the space $(\mathscr{X}_\pi^{(p)}, \|\cdot\|_\pi^{(p)})$ is a Banach space.

p-th variation as a limsup and variation index

Variation index

Let $\Pi([0,T])$ be the set of all refining partition sequences on $[0,T]$ with vanishing mesh.

Definition

The *variation index* of $x \in C^0([0,T],\mathbb{R})$ along any $\pi \in \Pi([0,T])$ is

$$p^\pi(x) := \inf\left\{p \geq 1 : \limsup_{n\to\infty} [x]^{(p)}_{\pi^n}(T) < \infty\right\}$$
$$= \inf\left\{p \geq 1 : \|x\|^{(p)}_{\pi} < \infty\right\}.$$

Since we have

$$\limsup_{n\to\infty} [x]^{(q)}_{\pi^n}(T) = \begin{cases} 0, & q > p^\pi(x), \\ \infty, & q < p^\pi(x), \end{cases}$$

the variation index can be a measure for 'roughness' of x along π. E.g. for a Brownian motion B, we have $p^\pi(B) = 2$ almost surely for any $\pi \in \Pi([0,T])$.

Variation index $p^\pi(x)$ is π-dependent

Proposition (Freedman, 1983)

Let $x \in C^0([0,T],\mathbb{R})$. There exists a sequence of partitions $\pi = (\pi^n)$ such that $[x]^{(2)}_\pi(T) = \limsup_{n\to\infty}[x]^{(2)}_{\pi^n}(T) = 0$.

This particular sequence π is of Lebesgue-type (partitioning the range of x).

It can be generalized to:

Proposition

For any $p > 1$ and $x \in C^0([0,T],\mathbb{R})$, there exists a sequence of partitions $\pi = (\pi^n)$ s.t. $[x]^{(p)}_\pi(T) = \limsup_{n\to\infty}[x]^{(p)}_{\pi^n}(T) = 0$.

Variation index $p^\pi(x)$ is π-dependent

From the previous result with a fact from the rough path theory,

Theorem

For any $x \in C^0([0,T])$, we have

$$\inf\{p^\pi(x) : \pi \in \Pi([0,T])\} = 1.$$

Moreover, for any $x \in C^{0,\alpha}([0,T])$, we have

$$\sup\{p^\pi(x) : \pi \in \Pi([0,T])\} = \frac{1}{\alpha}.$$

Thus, for any $x \in C^{0,\alpha}([0,T])$ and $\pi \in \Pi([0,T])$, we have

$$\boxed{1 \leq p^\pi(x) \leq \frac{1}{\alpha}}$$

Question: how can we characterize $p^\pi(x)$ of a function x along π?

Schauder representation along a general partition sequence

Previous results:

• (Cont & Das, 2022) Quadratic variation along refining partitions: Construction and examples, *J. Math. Anal. Appl.*

• (Bayraktar, Das & Kim, 2024) Hölder regularity and roughness: Construction and examples, *Bernoulli*.

Construction of Brownian Motion

Our first step is to give a more convenient representation for the processes $B^{(n)}$, $n = 0, 1, \ldots$. We define the *Haar functions* by $H_1^{(0)}(t) = 1$, $0 \le t \le 1$, and for $n \ge 1$, $k \in I(n)$,

$$H_k^{(n)}(t) = \begin{cases} 2^{(n-1)/2}, & \frac{k-1}{2^n} \le t < \frac{k}{2^n}, \\ -2^{(n-1)/2}, & \frac{k}{2^n} \le t < \frac{k+1}{2^n}, \\ 0, & \text{otherwise.} \end{cases}$$

We define the *Schauder functions* by

$$S_k^{(n)}(t) = \int_0^t H_k^{(n)}(u)\,du, \quad 0 \le t \le 1, n \ge 0, k \in I(n).$$

Note that $S_1^{(0)}(t) = t$, and for $n \ge 1$ the graphs of $S_k^{(n)}$ are little tents of height $2^{-(n+1)/2}$ centered at $k/2^n$ and nonoverlapping for different values of $k \in I(n)$. It is clear that $B_t^{(0)} = \xi_1^{(0)} S_1^{(0)}(t)$, and by induction on n, it is easily verified that

$$B_t^{(n)}(\omega) = \sum_{m=0}^{n} \sum_{k \in I(m)} \xi_k^{(m)}(\omega) S_k^{(m)}(t), \quad 0 \le t \le 1, n \ge 0. \tag{3.2}$$

3.1 Lemma. *As $n \to \infty$, the sequence of functions $\{B_t^{(n)}(\omega); 0 \le t \le 1\}$, $n \ge 0$, given by (3.2) converges uniformly in t to a continuous function $\{B_t(\omega); 0 \le t \le 1\}$, for a.e. $\omega \in \Omega$.*

3.2 Theorem. *With $\{B_t^{(n)}\}_{n=1}^{\infty}$ defined by (3.2) and $B_t = \lim_{n\to\infty} B_t^{(n)}$, the process $\{B_t, \mathscr{F}_t^B; 0 \le t \le 1\}$ is a Brownian motion on $[0, 1]$.*

Sequence of partitions

A sequence of partitions π of $[0, T]$ is a sequence $(\pi^n)_{n\geq 0}$:

$$\pi^n = (0 = t_0^n < t_1^n < \cdots < t_{N(\pi^n)}^n = T),$$

with

$$\underline{\pi^n} := \inf_{i=0,\cdots,N(\pi^n)-1} |t_{i+1}^n - t_i^n|, \qquad |\pi^n| := \sup_{i=0,\cdots,N(\pi^n)-1} |t_{i+1}^n - t_i^n|$$

For a fixed $n \geq 0$, we denote

$$p(n,k) := \inf_j \{t_j^{n+1} \geq t_k^n\} \qquad \text{for } k = 0, \cdots, N(\pi^n) - 1,$$

then the next level partition π^{n+1} contains

$$0 \leq t_k^n = t_{p(n,k)}^{n+1} < t_{p(n,k)+1}^{n+1} < \cdots < t_{p(n,k+1)}^{n+1} = t_{k+1}^n \leq T.$$

Generalized Haar basis

Definition

The Haar basis associated with π is a collection of piecewise constant functions $\{\psi_{m,k,i}\}$, $m = 0, 1, \cdots, k = 0, \cdots, N(\pi^m) - 1, i = 1, \cdots, p(m, k+1) - p(m, k)$:

$$\psi_{m,k,i}(t) = \begin{cases} 0 & \text{if } t \notin \left[t^{m+1}_{p(m,k)}, t^{m+1}_{p(m,k)+i}\right) \\ \left(\frac{t^{m+1}_{p(m,k)+i} - t^{m+1}_{p(m,k)+i-1}}{t^{m+1}_{p(m,k)+i-1} - t^{m+1}_{p(m,k)}} \times \frac{1}{t^{m+1}_{p(m,k)+i} - t^{m+1}_{p(m,k)}} \right)^{\frac{1}{2}} & \text{if } t \in \left[t^{m+1}_{p(m,k)}, t^{m+1}_{p(m,k)+i-1}\right) \\ -\left(\frac{t^{m+1}_{p(m,k)+i-1} - t^{m+1}_{p(m,k)}}{t^{m+1}_{p(m,k)+i} - t^{m+1}_{p(m,k)+i-1}} \times \frac{1}{t^{m+1}_{p(m,k)+i} - t^{m+1}_{p(m,k)}} \right)^{\frac{1}{2}} & \text{if } t \in \left[t^{m+1}_{p(m,k)+i-1}, t^{m+1}_{p(m,k)+i}\right) \end{cases}$$

$\{\psi_{m,k,i}\}$ is an orthonormal family! $(< f, g >= \int f(t)g(t)dt)$

Generalized Schauder functions

Definition

The Schauder functions $e^{\pi}_{m,k,i}$ are obtained by integrating $\psi^{\pi}_{m,k,i}$:

$$e^{\pi}_{m,k,i}(t) = \int_0^t \psi_{m,k,i}(s)ds = \left(\int_{t^{m+1}_{p(m,k)}}^{t^{m+1}_{p(m,k)+i} \wedge t} \psi_{m,k,i}(s)ds \right) \mathbb{1}_{[t^m_k, t^{m+1}_{p(m,k)+i}]}(t).$$

$e^{\pi}_{m,k,i} : [0, T] \to \mathbb{R}$ are triangle-shaped, continuous; $e^{\pi}_{m,k,i}(t) =$

$$\begin{cases} 0 & \text{if } t \notin \left[t^{m+1}_{p(m,k)}, t^{m+1}_{p(m,k)+i}\right) \\ \left(\frac{t^{m+1}_{p(m,k)+i} - t^{m+1}_{p(m,k)+i-1}}{t^{m+1}_{p(m,k)+i-1} - t^{m+1}_{p(m,k)}} \times \frac{1}{t^{m}_{p(m,k)+i} - t^{m}_{p(m,k)}} \right)^{\frac{1}{2}} \times (t - t^{m+1}_{p(m,k)}) & \text{if } t \in \left[t^{m+1}_{p(m,k)}, t^{m+1}_{p(m,k)+i-1}\right) \\ \left(\frac{t^{m+1}_{p(m,k)+i-1} - t^{m+1}_{p(m,k)}}{t^{m+1}_{p(m,k)+i} - t^{m+1}_{p(m,k)+i-1}} \times \frac{1}{t^{m+1}_{p(m,k)+i} - t^{m+1}_{p(m,k)}} \right)^{\frac{1}{2}} \times (t^{m+1}_{p(m,k)+i} - t) & \text{if } t \in \left[t^{m+1}_{p(m,k)+i-1}, t^{m+1}_{p(m,k)+i}\right) \end{cases}$$

Triadic Schauder functions

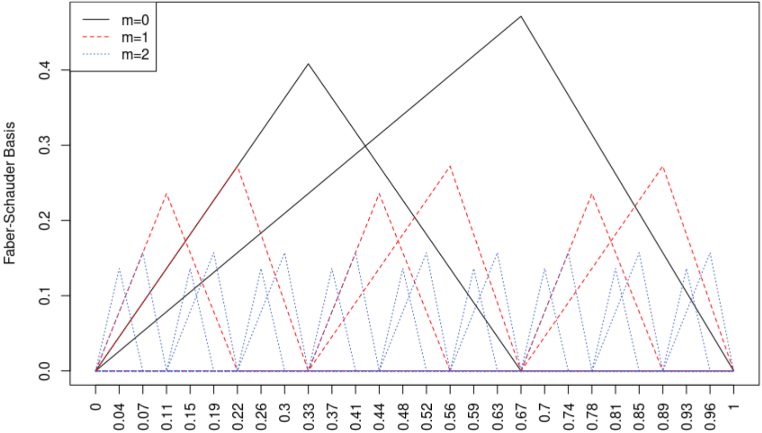

Figure 1: Schauder functions $e^{\pi}_{m,k}$ for $m = 0, 1, 2$ along triadic partition

Schauder representation of a continuous function

Proposition

Let π be a finitely refining sequence of partitions of $[0, T]$. Any $x \in C^0([0,T],\mathbb{R})$ has unique Schauder representation along π:

$$x(t) = x(0) + \big(x(T) - x(0)\big)t + \sum_{m=0}^{\infty} \sum_{k \in I_m} \theta^{x,\pi}_{m,k} e^{\pi}_{m,k}(t).$$

If support of $e^{\pi}_{m,k}$ is $[t_1^{m,k}, t_3^{m,k}]$ and its maximum is attained at $t_2^{m,k}$, then the coefficient $\theta^{x,\pi}_{m,k}$ has an explicit representation:

$$\theta^{x,\pi}_{m,k} = \frac{\Big(x(t_2^{m,k}) - x(t_1^{m,k})\Big)(t_3^{m,k} - t_2^{m,k}) - \Big(x(t_3^{m,k}) - x(t_2^{m,k})\Big)(t_2^{m,k} - t_1^{m,k})}{\sqrt{(t_2^{m,k} - t_1^{m,k})(t_3^{m,k} - t_2^{m,k})(t_3^{m,k} - t_1^{m,k})}}.$$

Refining (nested) partitions, $\pi = (\pi^n)_{n\in\mathbb{N}}$, $\pi^1 \subseteq \pi^2 \subseteq \cdots$

Definition (Finitely refining sequence of partitions)

A refining sequence π of $[0, T]$ is said to be finitely refining, if $|\pi^n| \to 0$ and $\exists M < \infty$ such that number of partition points of π^{n+1} within any two consecutive partition points of π^n is always bounded above by M, irrespective of $n \in \mathbb{N}$.

Recall: A sequence of partitions π of $[0, T]$ is a sequence $(\pi^n)_{n\geq 0}$:

$$\pi^n = (0 = t_0^n < t_1^n < \cdots < t_{N(\pi^n)}^n = T),$$

with

$$\underline{\pi^n} := \inf_{i=0,\cdots,N(\pi^n)-1} |t_{i+1}^n - t_i^n|, \qquad |\pi^n| := \sup_{i=0,\cdots,N(\pi^n)-1} |t_{i+1}^n - t_i^n|$$

Definition (Balanced sequence of partitions)

A sequence π is said to be balanced, if there exists a constant $c > 1$ satisfying

$$\frac{|\pi^n|}{\underline{\pi^n}} \leq c, \qquad \forall n \in \mathbb{N}.$$

Refining (nested) partitions, $\pi = (\pi^n)_{n\in\mathbb{N}}$, $\pi^1 \subseteq \pi^2 \subseteq \cdots$

Definition (Complete refining sequence of partitions)

A refining sequence π is said to be complete refining, if there exists positive constants a and b satisfying

$$1 + a \leq \frac{|\pi^n|}{|\pi^{n+1}|} \leq b, \qquad \forall n \in \mathbb{N}.$$

Definition (Convergent refining sequence of partitions)

A refining sequence π is said to be convergent refining, if there exists a constant $r > 1$ satisfying

$$\lim_{n\to\infty} \frac{|\pi^n|}{|\pi^{n+1}|} = r.$$

Convergent refining $\implies$ Complete refining

Main results

Schauder coefficients and p-th variation

Consider a Schauder representation of $x \in C^0([0,T],\mathbb{R})$ along π:

$$x(t) := \sum_{n=0}^{\infty} \sum_{k \in I_n} \theta_{n,k}^{x,\pi} e_{n,k}^{\pi}(t), \qquad t \in [0,T].$$

Theorem

For $p > 1$ and a balanced, convergent refining partition sequence π, let

$$\xi_n^{x,\pi,p} := |\pi^n|^{\frac{p}{2}} \left(\sum_{k \in I_n} |\theta_{n,k}^{x,\pi}|^p \right), \qquad \forall\, n \geq 0.$$

Then, we have $\quad \limsup_{n\to\infty} \xi_n^{x,\pi,p} < \infty \iff \limsup_{n\to\infty} [x]_{\pi^n}^{(p)} < \infty.$

Here, the expression $\xi_n^{\pi,(p)}$ depends only on the ℓ^p-norm of the n-th level Schauder coefficients $(\theta_{n,k}^{x,\pi})_{k \in I_n} = (\theta_{n,0}^{x,\pi}, \theta_{n,1}^{x,\pi}, \cdots, \theta_{n,|I_n|-1}^{x,\pi})$.

Characterization of variation index

Corollary

For any $x \in C^0([0,T],\mathbb{R})$ *and a balanced, convergent refining* π,

$$p^\pi(x) := \inf\left\{p \geq 1 : \limsup_{n\to\infty} [x]^{(p)}_{\pi^n}(T) < \infty\right\}$$
$$= \inf\left\{p \geq 1 : \limsup_{n\to\infty} \xi_n^{x,\pi,p} < \infty\right\}.$$

In other words,

$$\boxed{\limsup_{n\to\infty} \xi_n^{x,\pi,p} < \infty \iff x \in \mathscr{X}_\pi^{(p)}}$$

where $\mathscr{X}_\pi^{(p)} := \{x \in C^0([0,T],\mathbb{R}) : \|x\|_\pi^{(p)} < \infty\}$ is a Banach space with the norm $\|x\|_\pi^{(p)} := |x(0)| + \sup_{n\geq 0}\left([x]^{(p)}_{\pi^n}(T)\right)^{\frac{1}{p}}$.

Schauder coefficients and p-th variation

$$\boxed{\limsup_{n\to\infty} \xi_n^{x,\pi,p} < \infty \iff x \in \mathscr{X}_\pi^{(p)}}$$

$$\xi_n^{x,\pi,p} := |\pi^n|^{\frac{p}{2}} \left(\sum_{k\in I_n} |\theta_{n,k}^{x,\pi}|^p \right), \qquad \forall\, n \geq 0.$$

Thus, the finiteness of ℓ^p-norm of $(\theta_{n,k}^{x,\pi})_{k\in I_n}$ for each $n \geq 0$ is closely related to the finiteness of p-th variation of x.

On the other hand, the finiteness of ℓ^∞-norm of $(\theta_{n,k}^{x,\pi})_{n\geq 0, k\in I_n}$ is connected to the Hölder regularity of x, due to CIESIELSKI.

Schauder coefficients and Hölder exponent

Recall the α-Hölder norm of $x \in C^0([0,T],\mathbb{R})$:

$$\|x\|_{C^{0,\alpha}} := \|x\|_\infty + |x|_{C^{0,\alpha}} := \sup_{t\in[0,T]} |x(t)| + \sup_{\substack{t,s\in[0,T]\\ t\neq s}} \frac{|x(t)-x(s)|}{|t-s|^\alpha}.$$

Theorem (Ciesielski, 1960)

Let $\mathbb{T}$ be the dyadic partition sequence. Then, we have

$$x \in C^{0,\alpha}([0,T],\mathbb{R}) \iff \sup_{m,k}\left(2^{(m+1)(\alpha-\frac{1}{2})}|\theta^{x,\mathbb{T}}_{m,k}|\right) < \infty.$$

Moreover, the mapping

$$\begin{aligned} T^{\mathbb{T}}_\alpha : C^{0,\alpha}([0,T],\mathbb{R}) &\longrightarrow \quad \ell^\infty(\mathbb{R}) \\ x \quad &\longmapsto \left\{2^{(m+1)(\alpha-\frac{1}{2})}|\theta^{x,\mathbb{T}}_{m,k}|\right\}_{m,k}. \end{aligned}$$

is an isomorphism.

A recent generalization to general partition sequences

Theorem (Bayraktar, Das & Kim, 2023)

For any balanced and complete refining partition sequence π, we have

$$x \in C^{0,\alpha}([0,T],\mathbb{R}) \iff \sup_{m,k}\left(|\theta^{x,\pi}_{m,k}||\pi^{m+1}|^{\frac{1}{2}-\alpha}\right) < \infty,$$

and the following mapping is also an isomorphism:

$$\begin{aligned} T^{\pi}_{\alpha} : C^{0,\alpha}([0,T],\mathbb{R}) &\longrightarrow \quad \ell^{\infty}(\mathbb{R}) \\ x \quad &\longmapsto \left\{|\pi^{m+1}|^{\frac{1}{2}-\alpha}|\theta^{x,\pi}_{m,k}|\right\}_{m,k}. \end{aligned}$$

Also, we have the bounds

$$\frac{1}{\max\left(2M\sqrt{c}K_1^{\alpha}+2MK_2^{\alpha},\ MK_2^{\alpha}|\pi^1|^{\alpha}\right)}\|x\|_{C^{0,\alpha}} \le \sup_{m,k}\left(|\theta^{x,\pi}_{m,k}||\pi^{m+1}|^{\frac{1}{2}-\alpha}\right) \le 2(\sqrt{c})^3\|x\|_{C^{0,\alpha}},$$

with $K_1^{\alpha} := \frac{1}{1-(1+a)^{\alpha-1}}$ *and* $K_2^{\alpha} := \frac{1}{1-(1+a)^{-\alpha}}$.

A recent generalization to general partition sequences

We may arrange Schauder coefficients in an infinite-dimensional matrix:

$$\Theta^{x,\pi} = \begin{bmatrix} \theta^{x,\pi}_{0,0} & \theta^{x,\pi}_{0,1} & \cdots & \theta^{x,\pi}_{0,|I_0|-1} & 0 & 0 & 0 & \cdots \\ \theta^{x,\pi}_{1,0} & \theta^{x,\pi}_{1,1} & & \cdots & \theta^{x,\pi}_{1,|I_1|-1} & 0 & 0 & \cdots \\ \theta^{x,\pi}_{2,0} & \theta^{x,\pi}_{2,1} & & & \cdots & \theta^{x,\pi}_{2,|I_2|-1} & 0 & \cdots \\ \vdots & \vdots & & & & & & \end{bmatrix}$$

We also define a diagonal matrix D^{π}_{α} whose (m,m)-th entry is $|\pi^{m+1}|^{\frac{1}{2}-\alpha}$.

Then, we have

$$\sup_{m,k} \left(|\theta^{x,\pi}_{m,k}| |\pi^{m+1}|^{\frac{1}{2}-\alpha} \right) = \| D^{\pi}_{\alpha} \Theta^{x,\pi} \|_{sup},$$

where $\|A\|_{sup} := \sup_{m,k\geq 0} |A_{m,k}|$ is the supremum norm for matrices.

A recent generalization to general partition sequences

Therefore, the previous isomorphism can be reformulated as...

Theorem (Bayraktar, Das & Kim, 2023)

For any balanced, complete refining partition sequence π *and* $\alpha \in (0,1)$, *the mapping*

$$\begin{array}{rcl} T^{\pi}_{\alpha} : \left(C^{0,\alpha}([0,T],\mathbb{R}), \|\cdot\|_{C^{0,\alpha}}\right) & \longrightarrow & \left(\mathscr{M}^{\alpha}_{\pi}, \|\cdot\|^{\alpha}_{sup}\right) \\ x & \longmapsto & \Theta^{x,\pi} \end{array}$$

is an isomorphism, where

$$\mathscr{M}^{\alpha}_{\pi} := \{A \in \mathscr{M} : A_{m,k} = 0 \text{ for } k > |I_m| \text{ and } \|A\|^{\alpha}_{sup} < \infty\},$$

$$\|A\|^{\alpha}_{sup} := \|D^{\pi}_{\alpha} A\|_{sup}.$$

Schauder coefficients and p-th variation

For any $p \in (1, \infty)$, recall the main result

$$\boxed{\limsup_{n\to\infty} \xi_n^{x,\pi,p} < \infty \iff x \in \mathscr{X}_\pi^{(p)}}$$

$$\xi_n^{x,\pi,p} := |\pi^n|^{\frac{p}{2}} \left(\sum_{k \in I_n} |\theta_{n,k}^{x,\pi}|^p \right), \qquad \forall\, n \geq 0.$$

We define another diagonal matrix $Q^\pi_{(p)}$ whose (m,m)-th entry is $|\pi^m|^{\frac{p}{2}}$.

Moreover, we define a subspace of infinite-dimensional matrices

$$\mathscr{M}_\pi^{(p)} := \{A \in \mathscr{M} : A_{m,k} = 0 \text{ for } k > |I_m| \text{ and } \|A\|_{(p)} < \infty\},$$

$$\|A\|_{(p)} := \|(Q^\pi_{(p)} A)^T\|_{p,\infty}, \qquad \|B\|_{p,\infty} := \sup_{k\geq 0} \left(\sum_{m \geq 0} |B_{m,k}|^p \right)^{\frac{1}{p}}.$$

Schauder coefficients and p-th variation

Theorem

For any $\alpha \in (0,1)$, $p \in (1, \frac{1}{\alpha}]$, *the space*

$$\left(C^{0,\alpha}([0,T],\mathbb{R}) \cap \mathscr{X}_\pi^{(p)}, \ \ \|\cdot\|_{C^{0,\alpha}} + \|\cdot\|_\pi^{(p)} \right)$$

is a Banach space.

Furthermore, if π *is balanced and convergent refining, then the mapping*

$$T^\pi_{\alpha,(p)} : \left(C^{0,\alpha}([0,T],\mathbb{R}) \cap \mathscr{X}_\pi^{(p)}, \ \|\cdot\|_{C^{0,\alpha}} + \|\cdot\|_\pi^{(p)} \right) \longrightarrow \left(\mathscr{M}_\pi^\alpha \cap \mathscr{M}_\pi^{(p)}, \ \|\cdot\|_{sup}^\alpha + \|\cdot\|_{(p)} \right)$$
$$x \longmapsto \Theta^{x,\pi}$$

is an isomorphism.

This provides a characterization of α-Hölder functions with finite p-th variation along π.