Mild to classical solutions for XVA equations under stochastic volatility

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A financial market model with default

We aim to evaluate a derivative contract between an investor $\mathcal{I}$ and a counterparty $\mathcal{C}$ in a financial market under

- default risk,
- collateralisation and
- funding costs and benefits.

To this end, we derive a valuation equation based on default-free information only and characterise its solutions, the pre-default value processes.

By focusing on a stochastic volatility model, we will reach a parabolic semilinear PDE that establishes a direct relation between pre-default value processes and mild solutions.

In what follows, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $T > 0$ stand for the maturity of the contract. Further, let

$$\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]} \quad \text{and} \quad \mathbb{G} = (\mathcal{G}_t)_{t \in [0,T]}$$

be two filtrations of $\mathcal{F}$ that model the temporal developments of the default-free information and the whole available information on an underlying financial market, respectively.

We use two $[0, T] \cup \{\infty\}$-valued random variables $\tau_{\mathcal{I}}$ and $\tau_{\mathcal{C}}$ to model the respective default times of $\mathcal{I}$ and $\mathcal{C}$. Then

$$\tau := \tau_{\mathcal{I}} \wedge \tau_{\mathcal{C}}$$

stands for the time of a party to default first.

By using the smallest filtration under which $\tau_{\mathcal{I}}$ and $\tau_{\mathcal{C}}$ are stopping times, we require that

$$\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F}_t \vee \sigma\big(\mathbb{1}_{\{\tau_i \leq s\}} : i \in \{\mathcal{I}, \mathcal{C}\}, \, s \in [0, t]\big)$$

for all $t \in [0, T]$. Thus, the available market information may fail to give full insight into $\tau_{\mathcal{I}}$ or $\tau_{\mathcal{C}}$ and it could yield no or only partial knowledge about τ.

We assume that the distributions of $\tau_{\mathcal{I}}$ and $\tau_{\mathcal{C}}$ admit at most one atom, which is at infinity, and both parties cannot default at the same time. That is,

$$\mathbb{P}(\tau_{\mathcal{I}} = t) = \mathbb{P}(\tau_{\mathcal{C}} = t) = \mathbb{P}(\tau_{\mathcal{I}} = \tau_{\mathcal{C}}, \, \tau < \infty) = 0 \qquad \text{(C)}$$

for all $t \in [0, T]$. However, both entities may not default at all. So, we allow for $\mathbb{P}(\tau = \infty) \in [0, 1]$.

Example (Hitting times involving a gamma distribution)

Let λ^i be an $\mathbb{F}$-progressively measurable process and ξ_i be a gamma distributed random variable such that $\lambda^i, \xi_i > 0$ and

$$\tau_i = \inf\left\{ t \in [0,T] \,\middle|\, \int_0^t \lambda_s^i \, ds \geq \xi_i \right\}$$

for $i \in \{\mathcal{I}, \mathcal{C}\}$. Then, under verifiable assumptions, the conditions in (C) on the distribution of $\tau_{\mathcal{I}}$ and $\tau_{\mathcal{C}}$ hold and

$$\mathbb{P}(\tau \in B) = \int_{B \cap [0,T]} \varphi_\tau(s) \, ds + \left(1 - \int_0^T \varphi_\tau(s) \, ds\right) \delta_\infty(B)$$

for any Borel set B in $[0,T] \cup \{\infty\}$ and some explicitly determined measurable integrable function $\varphi_\tau : [0,T] \to [0,\infty]$.

By $\mathcal{S}$ and $\tilde{\mathcal{S}}$ we denote the linear space of all (real-valued) processes that are adapted to $\mathbb{F}$ and $\mathbb{G}$, respectively.

Next, the measurable integrable function $r : [0, T] \to \mathbb{R}$ is the instantaneous risk-free interest rate and

$$D_{s,t}(r) := \exp\left(- \int_s^t r(\tilde{s})\, d\tilde{s} \right)$$

is the discount factor from time $s \in [0, T]$ to $t \in [s, T]$. Put differently, $D_{s,t}(r)$ is the required amount to invest risk-free at time s, in order to receive 1 unit of cash at time t.

We let $\mathbb{Q}$ be an equivalent local martingale measure. That is, $\mathbb{P} \sim \mathbb{Q}$ and the discounted price process of the only traded risky asset is an $\mathbb{G}$-local martingale under $\mathbb{Q}$.

We derive an equation for the value process $\tilde{\mathcal{V}} \in \tilde{\mathcal{S}}$ of the derivative contract between $\mathcal{I}$ and $\mathcal{C}$ under $\mathbb{Q}$.

In the end, however, we seek a valuation that does not involve any knowledge of the default of either of the two parties.

At the same time, the valuation equation for $\tilde{\mathcal{V}}$ includes quantities that merely depend on its pre-default part in the following sense.

Let $G(\tau)$ denote an $\mathbb{F}$-survival process of τ under $\mathbb{Q}$, which is an $[0,1]$-valued supermartingale under $\mathbb{Q}$ such that

$$\mathbb{Q}(\tau > t|\mathcal{F}_t) = G_t(\tau) \quad \text{a.s. for all } t \in [0, T].$$

Further, a process $\tilde{X}$ is called integrable up to time τ if the process $[0, T] \times \Omega \to \mathbb{R}$, $(t, \omega) \mapsto \tilde{X}_t(\omega)\mathbb{1}_{\{\tau > t\}}(\omega)$ is integrable.

We refine a classical result as follows.

Pre-default versions

A process $\tilde{X} \in \tilde{\mathcal{S}}$ is integrable up to time τ if and only if there is $X \in \mathcal{S}$ such that $XG(\tau)$ is integrable and $X_s = \tilde{X}_s$ a.s. on $\{\tau > s\}$ for all $s \in [0, T]$. In this case,

$$X_s G_s(\tau) = \mathbb{E}_{\mathbb{Q}}[\tilde{X}_s \mathbb{1}_{\{\tau > s\}} | \mathcal{F}_s] \quad \text{a.s.}$$

for all $s \in [0, T]$. If in addition $G_s(\tau) > 0$ a.s. for all $s \in [0, T]$, then X is unique up to a modification.

We shall call X a pre-default version of $\tilde{X}$.

Thereby, $G_s(\tau) > 0$ a.s. for all $s \in [0, T]$ implies that the probability that neither $\mathcal{I}$ nor $\mathcal{C}$ defaults before any time is positive:

$$\mathbb{Q}(\tau > s) > 0 \quad \text{for any } s \in [0, T].$$

In the spirit of the preceding result, we introduce valuation based on default-free information only.

More precisely, we analyse any pre-default values process $\mathcal{V}$ defined as pre-default version of $\tilde{\mathcal{V}}$, which in turn should be integrable up to time τ.

The discounted cash flows, costs and benefits

Let us summarise the cash flows, costs and benefits that may impact the value of the contract between $\mathcal{I}$ and $\mathcal{C}$.

1. **The contractual cash flows** depend on a payoff functional and a dividend-paying risky asset:

$$\mathrm{CF}_s^{\mathrm{con}} := D_{s,T}(r)\Phi(S)\mathbb{1}_{\{\tau>T\}} + \int_s^{T\wedge\tau} D_{s,t}(r)\pi_t\,dt. \tag{1}$$

2. **The costs and benefits of a collateral account**, subject to the collateral remuneration rate, are of the form

$$\mathrm{C}_s^{\mathrm{col}}(\mathcal{V}) := \int_s^{T\wedge\tau} D_{s,t}(r)(c_t(\mathcal{V}) - r(t))C_t(\mathcal{V})\,dt, \tag{2}$$

where $c_t(\mathcal{V}) := c_t^{(+)}\mathbb{1}_{\{C_t(\mathcal{V})>0\}} + c_t^{(-)}\mathbb{1}_{\{C_t(\mathcal{V})<0\}}$.

3. **The costs and benefits of a funding account** based on the interest rates for borrowing and lending are

$$\mathrm{C}^{\mathrm{fun}}_s(\tilde{\mathcal{V}}) := \int_s^{T\wedge\tau} D_{s,t}(r)(\tilde{f}_t(\tilde{\mathcal{V}}) - r(t))\tilde{F}_t(\tilde{\mathcal{V}})\,dt \qquad (3)$$

with $\tilde{f}_t(\tilde{\mathcal{V}}) := \tilde{f}_t^{(+)}\mathbb{1}_{\{\tilde{F}_t(\tilde{\mathcal{V}})>0\}} + {}_{-}\tilde{f}_t\mathbb{1}_{\{\tilde{F}_t(\tilde{\mathcal{V}})<0\}}$.

4. **The cash flows arising on the default** of $\mathcal{I}$ or $\mathcal{C}$ are computed with the residual value of the claim:

$$\mathrm{CF}^{\mathrm{def}}_s(\mathcal{V}) := D_{s,\tau}(r)\varepsilon_\tau(\mathcal{V}) \qquad (4)$$

on $\{s < \tau < T\}$ and $\mathrm{CF}^{\mathrm{def}}_s(\mathcal{V}) := 0$ on the complement of this set.

Under mild path and integrability conditions, we require that the value process $\tilde{\mathcal{V}}$ of the derivative contract satisfies the valuation equation

$$\tilde{\mathcal{V}}_s = \mathbb{E}_{\mathbb{Q}}\big[\mathrm{CF}_s^{\mathrm{con}} - \mathrm{C}_s^{\mathrm{col}}(\mathcal{V}) - \mathrm{C}_s^{\mathrm{fun}}(\tilde{\mathcal{V}}) + \mathrm{CF}_s^{\mathrm{def}}(\mathcal{V})\big|\mathcal{G}_s\big] \qquad (5)$$

a.s. for all $s \in [0, T]$. In particular, $\tilde{\mathcal{V}}_T = \Phi(S)$ a.s. must hold.

That is, $\tilde{\mathcal{V}}_s$ should agree with the conditional expectation of the sum of the net present values of all cash flows, costs and benefits relative to the current available market information.

This hypothesis is based on an adjusted cash flow approach and the article [4] shows in a rather general setting that a replication approach results in the same nonlinear valuation formula.

Further, this implicit conditional representation refines the valuation equation in the work [5] of Brigo, Francischello and Pallavicini.

As a result, we obtain a valuation equation involving default-free information only for the pre-default version $\mathcal{V}$ of $\tilde{\mathcal{V}}$.

Pre-default valuation (Brigo, Graceffa and K., 2024)

Under weak conditions, (5) is satisfied if and only if

$$\mathcal{V}_s G_s(\tau) = \mathbb{E}_{\mathbb{Q}}\left[D_{s,T}(r)\Phi(S)G_T(\tau) + \int_s^T D_{s,t}(r)\, dA_t(\mathcal{V}) \,\middle|\, \mathcal{F}_s \right]$$

a.s. for any $s \in [0, T]$, where

$$A_t(\mathcal{V}) := \int_0^t \mathrm{B}_s^{(0)}(\mathcal{V}) G_s(\tau)\, ds - \int_0^t \varepsilon_s(\mathcal{V}) G_s(\tau)\, dG_s(\tau)$$

with $\mathrm{B}_s^{(0)}(\mathcal{V}) := \pi_t - (c_t(\mathcal{V}) - r(t))C_t(\mathcal{V}) - (f_t(\mathcal{V}) - r(t))F_t(\mathcal{V})$.

Martingale characterisation of pre-default valuation (Brigo, Graceffa and K., 2024)

Under mild conditions, $\mathcal{V}$ is a pre-default value process if and only if $M^{\mathcal{V}} \in \mathcal{S}$ defined via

$$M_t^{\mathcal{V}} := D_{0,t}(r)\mathcal{V}_t G_t(\tau) + \int_0^t D_{0,s}(r)\, dA_s(\mathcal{V})$$

is an $\mathbb{F}$-martingale under $\mathbb{Q}$ and $\mathcal{V}_T = \Phi(S)$ a.s. on $\{G_T(\tau) > 0\}$.

Characterisation of pre-default value semimartingales (Brigo, Graceffa and K., 2024)

Under weak conditions, the continuous $\mathbb{F}$-semimartingale $\mathcal{V}$ is a pre-default value process if and only if $\mathbb{E}_{\mathbb{Q}}[|\mathcal{V}_0|] < \infty$ and

$$\mathcal{V}_s = \Phi(S) + \int_s^T \left(\mathrm{B}_t^{(0)}(\mathcal{V}) - r(t)\mathcal{V}_t\right) dt \\ - \int_s^T \frac{\varepsilon_t(\mathcal{V}) - \mathcal{V}_t}{G_t(\tau)} \, d\, G_t(\tau) - \int_s^T \frac{D_{0,t}(-r)}{G_t(\tau)} \, dM_t$$

for all $s \in [0, T]$ a.s. and some continuous $\mathbb{F}$-martingale M under $\mathbb{Q}$. In this case, $M - M_0 = M^{\mathcal{V}} - \mathcal{V}_0$ a.s.

A stochastic volatility model

Next, we suppose that the usual conditions hold and $\hat{W}$ and $\tilde{W}$ are two $\mathbb{F}$-Brownian motions with covariation

$$\langle \hat{W}, \tilde{W} \rangle = \int_0^{\cdot} \rho(s)\, ds \quad \text{a.s.}$$

Let us impose the following dynamics on the price process S of the only risky asset and its squared volatility process V:

$$\begin{aligned} dS_t &= b(t)S_t\, dt + \sqrt{V_t} S_t\, d\hat{W}_t \\ dV_t &= \big(k(t) - l_0(t)V_t - l(t)V_t^{\alpha}\big)\, dt + \lambda(t)V_t^{\beta}\, d\tilde{W}_t \end{aligned} \tag{6}$$

for $t \in [0, T]$ with initial condition $(S_0, V_0) = (s_0, v_0)$ a.s., where $\alpha \geq 1$ and $\beta \geq 1/2$.

From a pathwise uniqueness and a strong existence result in [2] and a positivity condition we draw the following conclusion.

Power diffusion as squared volatility (Brigo, Graceffa and K., 2024)

Let b, k, l_0, l, λ be bounded, $k, l \geq 0$ and $\lambda^2 \leq 2k$. Then pathwise uniqueness for the SDE (6) holds and there is a unique strong solution (S, V) satisfying

$$S > 0, \quad V > 0 \quad \text{and} \quad (S_0, V_0) = (s_0, v_0) \quad \text{a.s.}$$

Further, $\sup_{t \in [0,T]} |X_t|$ and V are integrable, where $X := \log(S)$.

Example (Established models in the literature)

For $l = 0$ and $l_0 > 0$ we recover the dynamics

$$dV_t = \big(k(t) - l_0(t)V_t\big)\,dt + \lambda(t)V_t^{\beta}\,d\tilde{W}_t, \quad \text{for } t \in [0, T]$$

in time-dependent versions of the following option pricing models:

(1) The Heston model for $\beta = 1/2$. There, l_0 is the mean reversion speed, k/l_0 is the mean reversion level and the same positivity condition $\lambda \leq 2k$ applies.

(2) The Garch diffusion model for $\beta = 1$. Similarly, l_0 is the mean reversion speed and k/l_0 the mean reversion level.

The logarithmised pre-default valuation PDE

Let us combine the market model with the stochastic volatility model under the assumption that $\Phi(S) = \phi(S_T)$ and

$$\mathrm{B}_t^{(0)}(\mathcal{V}) - r(t)\mathcal{V}_t - \frac{\dot{G}_t(\tau)}{G_t(\tau)}(\varepsilon_t(\mathcal{V}) - \mathcal{V}_t) = \hat{B}(t, S_t, V_t, \mathcal{V}_t)$$

for all $t \in [0, T]$ and each continuous $\mathcal{V} \in \mathcal{S}$ and certain functions

$$\phi :]0, \infty[\to \mathbb{R}_+ \quad \text{and} \quad \hat{B} : [0, T] \times]0, \infty[^2 \times \mathbb{R} \to \mathbb{R}.$$

Then we can explicitly construct a local martingale measure via Girsanov's theorem by proposing suitable market prices of risk.

Thus, for two $\mathbb{F}$-progressively measurable processes η and $\tilde{\eta}$ with square-integrable paths, we define a continuous $\mathbb{F}$-local martingale Z via

$$Z = \exp\left(- \int_0^{\cdot} \eta_s \, dW_s - \int_0^{\cdot} \tilde{\eta}_s \, d\tilde{W}_s - \frac{1}{2} \int_0^{\cdot} \eta_s^2 + \tilde{\eta}_s^2 \, ds \right),$$

where W is an $\mathbb{F}$-Brownian motion that is independent of $\tilde{W}$. Under the condition that

$$\mathbb{E}\big[Z_T \big] = 1,$$

Girsanov's theorem entails that this process induces an equivalent local martingale measure $\hat{\mathbb{P}}_{\eta,\tilde{\eta}}$ if and only if

$$(b - r)(t) = \sqrt{V_t}\big(\eta_t \sqrt{1 - \rho(t)^2} + \tilde{\eta}_t \rho(t)\big)$$

for a.e. $t \in [0, T]$ a.s.

For this reason, we propose to take the market prices of risk

$$\tilde{\eta}_t = \gamma\sqrt{V_t} \quad \text{and} \quad \eta_t = \left(\frac{(b-r)(t)}{\sqrt{V_t}} - \tilde{\eta}_t\rho(t)\right)\frac{1}{\sqrt{1-\rho(t)^2}}$$

for all $t \in [0, T]$ and fixed $\gamma \geq 0$.

Under certain conditions, by approximating η via simple processes, we infer from Lemma 35 in [6] that

$$\mathbb{E}[Z_T] = \mathbb{E}\bigg[\exp\bigg(-\gamma\int_0^T \sqrt{V_t}\, d\tilde{W}_t - \frac{1}{2}\gamma^2\int_0^T V_t\, dt\bigg)\bigg],$$

and for $\mathbb{E}[Z_T] = 1$ to hold, it suffices that $\exp((\gamma^2/2)\int_0^T V_t\, dt)$ is $\mathbb{P}$-integrable, by Novikov's condition.

In this case, we set $\mathbb{Q}_{V,\gamma} := \hat{\mathbb{P}}_{\eta,\tilde{\eta}}$ and for the log-price process $X = \log(S)$ we see that (X, V) solves the SDE

$$d\begin{pmatrix} X_t \\ V_t \end{pmatrix} = \begin{pmatrix} r(t) - \frac{1}{2}V_t \\ k(t) - l_0(t)V_t - l(t)V_t^{\alpha} - \gamma\lambda(t)V_t^{\frac{1}{2}+\beta} \end{pmatrix} dt + \begin{pmatrix} \sqrt{V_t}\sqrt{1-\rho(t)^2} & \sqrt{V_t}\rho(t) \\ 0 & \lambda(t)V_t^{\beta} \end{pmatrix} d\begin{pmatrix} W_t^{(\eta)} \\ W_t^{(\tilde{\eta})} \end{pmatrix}$$

for $t \in [0, T]$ under $\mathbb{Q}_{V,\gamma}$, where

$$W^{(\eta)} := W + \int_0^{\cdot} \eta_s \, ds \quad \text{and} \quad W^{(\tilde{\eta})} := \tilde{W} + \int_0^{\cdot} \tilde{\eta}_s \, ds$$

are two independent $\mathbb{F}$-Brownian motions under $\mathbb{Q}_{V,\gamma}$. In particular, as $\gamma = 0$ is feasible, there is an equivalent local martingale measure.

This induces a differential operator $\mathcal{L}$ on $C^{1,2,2}([0,T[\times\mathbb{R}\times]0,\infty[)$ with values in the linear space of all real-valued measurable functions by

$$\begin{aligned}\mathcal{L}(u)(\cdot,x,v) := &\left(r-\frac{1}{2}v\right)\frac{\partial u}{\partial x}(\cdot,x,v)\\ &+\left(k-l_0 v-lv^{\alpha}-\gamma\lambda v^{\frac{1}{2}+\beta}\right)\frac{\partial u}{\partial v}(\cdot,x,v)\\ &+\frac{1}{2}v\frac{\partial^2 u}{\partial x^2}u(\cdot,x,v)+\lambda v^{\frac{1}{2}+\beta}\rho\frac{\partial^2 u}{\partial x\partial v}(\cdot,x,v)\\ &+\frac{1}{2}\lambda^2 v^{2\beta}\frac{\partial^2 u}{\partial v^2}(\cdot,x,v).\end{aligned}$$

This formula is obtained by multiplying the diffusion coefficient of the preceding SDE with its transpose, as we readily recall.

Hence, we reached a parabolic semilinear PDE that establishes a direct relation to mild solutions.

Mild solutions as pre-default valuation functions (Brigo, Graceffa and K., 2024)

Under certain conditions, if $u \in C([0,T] \times \mathbb{R} \times]0,\infty[)$ is a mild solution to the PDE with terminal condition

$$\frac{\partial u}{\partial t}(t,x,v) + \mathcal{L}(u)(t,x,v) = -\hat{B}(t,e^x,v,u(t,x,v)) \qquad (7)$$

and $u(T,x,v) = \phi(e^x)$ for $(t,x,v) \in [0,T[\times \mathbb{R} \times]0,\infty[$, then $\mathcal{V} \in \mathcal{S}$ defined via

$$\mathcal{V}_t := u(t,X_t,V_t)$$

is a pre-default value process under $\mathbb{Q}_{V,\gamma}$.

Finally, [3] yields sufficient conditions for mild solutions to exist uniquely and to admit non-negative or positive values only.

Existence and uniqueness of pre-default valuation functions

Under verifiable conditions, there exists a unique bounded mild solution u_ϕ to the parabolic PDE (7) such that

$$u_\phi(T, x, \cdot) = \phi(e^x) \quad \text{for all } x \in \mathbb{R}.$$

Moreover, u_ϕ is non-negative if

$$\hat{B}(\cdot, e^x, v, 0) \geq 0 \quad \text{for all } x \in \mathbb{R} \text{ and } v > 0.$$

In this case, $u_\phi > 0$ follows from $\phi > 0$.

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Thank you for your attention!