Ergodicity properties of affine term structure models.

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based on joint papers with P. Jin , V. Mandrekar and B. Rüdiger

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$$dX_t = a(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t + dJ_t, \quad X_0 = x \geq 0,$$

where $a, \sigma > 0$, $\theta \geq 0$,

- Exponential ergodicity results for JCIR processes
- Exp. ergodicity results and Harris recurrence as well as density of BAJD
- Harris recurrence and calibration results for CIR process

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Using Ergodicity for calibration

Idea: Let π be the invariant measure of an ergodic process X_s.

$$\frac{1}{t}\int_0^t f(X_s)ds \to \int f(x)\pi(dx)\,; \quad \text{time average} \to \ \text{average w.r.t.}\pi$$

Remark:

- the process depends on some constant a,θ,σ, as well as the invariant measure.
- Relation between process and invariant measure is known by stochastic analysis.
- Time avarage can be observed by historical data
- The constants can be found by time average, hence for the invariant measure and hence for the process

Outline of this talk concerning CIR and CIR with jumps

- Affine processes and properties
- CIR process and strong ergodic properties
- Application for calibration of the parameter of the CIR process
- A particular JCIR: the BAJD. Strong ergodic properties
- Exponential ergodicity of JCIR

Jump-diffusion CIR

Consider

$$dX_t = a(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t + dJ_t, \quad X_0 = x \geq 0,$$

where $a, \sigma > 0$, $\theta \geq 0$, W_t is a 1-dimensional Brownian motion and J_t is a stoch. independent pure-jump Lévy process with its Lévy measure ν concentrating on $(0, \infty)$ and satisfying

$$\int_{(0,\infty)} (\xi \wedge 1)\nu(d\xi) < \infty.$$

Existence and uniqueness of strong solution follows from results of Fu and Li (2010)

Affine processes on $\mathbb{R}_+$

Def. (see e.g. [Duffie, Filipovic, Schachermeyer 2003],..) An affine process with values in $\mathbb{R}_+$ is a time homogeneous Markov Process $(X_t)_{t\geq 0}$ which characteristic function of X_t (given $X_0 = x$) are exponentially affine, i.e there exist $\phi : \mathbb{R}_+ \times i\mathbb{R} \to \mathbb{C}$, $\psi : \mathbb{R}_+ \times i\mathbb{R} \to \mathbb{C}$, s.th.

$$E_x\left[e^{uX_t}\right] = e^{\phi(t,u)+x\psi(t,u)},$$

Theorem [Duffie, Filipovic, Schachermeyer An. Appl. Prob 2003, or Keller -Ressel, Schachermeyer, Teichmann PTRF 2011] An affine process is a Feller process (in particular $P_t C_0(\mathbb{R}) \subset C_0(\mathbb{R})$).

Properties of affine processes:

Theorem: $\phi(t,u)$ and $\psi(t,u)$ satisfy the "semi-flow property":

$$\phi(t+s,u)=\phi(s,u)+\phi(t,\psi(s,u)),\quad \psi(t+s,u)=\psi(t,\psi(s,u))$$

, **Proof:** Let $f_u(x):=e^{ux}$

$$P_t f_u(x)=E_x\left[e^{uX_t}\right]=e^{\phi(t,u)+x\psi(t,u)}=e^{\phi(t,u)}f_{\psi(t,u)}(x),$$

$$P_{t+s}f_u(x)=P_tP_sf_u(x)=e^{\phi(s,u)}P_tf_{\psi(s,u)}(x)=e^{\phi(s,u)+\phi(t,\psi(s,u))}f_{\psi(t,\psi(s,u))}$$

$$P_{t+s}f_u(x)=e^{\phi(t+s,u)}f_{\psi(t+s,u)}(x)$$

Def. An affine process is called regular, if the derivatives

$$F(u) := \frac{\partial \phi}{\partial t}(t,u)|_{t=0+} \quad R(u) := \frac{\partial \psi}{\partial t}(t,u)|_{t=0+}$$

exist and are continuous at $u = 0$.

Corollary: If $(X_t)_{t \geq 0}$ is a regular affine process then "Riccati equations" hold:

$$\begin{cases} \partial_t \phi(t,u) = F\big(\psi(t,u)\big), & \phi(0,u) = 0, \\ \partial_t \psi(t,u) = R\big(\psi(t,u)\big), & \psi(0,u) = u, \end{cases}$$

Proof: Use the semiflow property.

Cox-Ingersoll-Ross model (or CIR model)

- 1985, by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross
- A model to describe the evolution of interest rates
- The CIR process X_t is given as the unique strong solution of the following stochastic differential equation

$$dX_t = (a\theta - aX_t)dt + \sigma\sqrt{|X_t|}dW_t, \quad X_0 = x \geq 0,$$

where $a, \theta, \sigma > 0$ are constants and
W_t is a 1-dimensional Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with $\mathcal{F}_t$ satisfying the usual conditions.

Cox-Ingersoll-Ross model (or CIR model)

- The CIR model has a density
- when the rate X_t gets close to zero, the diffusion coefficient $\sigma\sqrt{|X_t|}$ also becomes close to zero.
- The singularity of the diffusion coefficient at the origin implies that an initially nonnegative interest rate can never subsequently become negative. $X_t \geq 0$ if $X_0 \geq 0$.
- Due to the result of Yamada-Watanabe (1989), the above SDE has a unique strong solution.
- The drift $(a\theta - aX_t)$ ensures mean reversion of the interest rate towards the long-term value θ.

- As well known, the CIR process is an affine process in $\mathbb{R}_+$, i.e.

$$E[exp(uX_t)] = exp(\phi(t,u)) + x\psi(t,u))$$
$$\forall (t,u) \in \mathbb{R}_+ \times \{u \in \mathbb{C} : Re(u) < 0\}$$

where the functions $\phi(t,u)$ and $\psi(t,u)$ solve the Riccati equations

$$\begin{cases} \partial_t \phi(t,u) = F\big(\psi(t,u)\big), & \phi(0,u) = 0, \\ \partial_t \psi(t,u) = R\big(\psi(t,u)\big), & \psi(0,u) = u, \end{cases} \tag{1}$$

with F and R satisfies

$$F(u) = a\theta u \tag{2}$$
$$R(u) = \frac{\sigma^2 u^2}{2} - au. \tag{3}$$

Solving the generalized Riccati equations, we get

$$\phi(t,u) = -\frac{2a\theta}{\sigma^2}\log\left(1 - \frac{\sigma^2}{2a}u(1-e^{-at})\right)$$

$$\psi(t,u) = \frac{ue^{-at}}{1 - \frac{\sigma^2}{2a}u(1-e^{-at})}$$

Thus

$$E[exp(uX_t)] = \int_{\mathbb{R}_+} p(t,x,y)e^{uy}dy = exp(\phi(t,u) + x\psi(t,u))$$
$$\left(1 - \frac{\sigma^2}{2a}u(1-e^{-at})\right)^{-\frac{2a\theta}{\sigma^2}} \cdot \exp\left(\frac{xue^{-at}}{1-\frac{\sigma^2}{2a}u(1-e^{-at})}\right).$$

Duffie et al. constructed the CIR process as a Markov process on the canonical path space. We denote :
$\hat{\Omega}$: the canonical path space, namely $\hat{\Omega} = C\big([0,\infty);\mathbb{R}_+\big)$
$\hat{X}_t$: the canonical process on $\hat{\Omega}$.
$(\hat{\mathcal{F}}_t)_{t\geq 0}$: the filtration generated by $\hat{X}_t$, namely
$\hat{\mathcal{F}}_t := \sigma(\hat{X}_s, 0 \leq s \leq t)$ and $\hat{\mathcal{F}} := \sigma(\hat{X}_s, s \geq 0)$. The map

$$X : (\Omega, \mathcal{F}) \to (\hat{\Omega}, \hat{\mathcal{F}})$$

induces a measure $\hat{P}_x$ on $(\hat{\Omega}, \hat{\mathcal{F}})$, which is the law of the CIR process starting from x on the canonical path space.
$\Rightarrow$ The Markov process $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}_x, x \in \mathbb{R}_+)$ is a Feller process.

Feller regularity Property

Lemma

CIR process is a "regular Feller process" on $\mathbb{R}_+$.

Definition: Consider a one -dim Feller process X with distributions P_x. The process is said to be a "regular Feller process" if there exists a locally finite measure ρ and a continuous function $(t, x, y) \mapsto p_t(x, y) > 0$ such that

$$P_x\{X_t \in B\} = \int_B p_t(x, y)\rho(dy), \quad x \in \mathbb{R},\ B \in \mathbb{R}_+,\ t > 0.$$

Idea of Proof

- The supporting measure for the CIR process can not be the Lebesgue measure.
- The behavior of the transition density at $y = 0$ violates the regularity conditions because
  if $\frac{2a\theta}{\sigma^2} < 1$, $p(t, x, 0) := \lim_{y\to 0} p(t, x, y) = \infty$ and if $\frac{2a\theta}{\sigma^2} > 1$, then $p(t, x, 0) := \lim_{y\to 0} p(t, x, y) = 0$
- To overcome this difficulty, we define a new measure ρ on $\big(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)\big)$ as

$$\rho(dx) := h(x)dx,$$

where

$$h(x) = \begin{cases} x^{\frac{2a\theta}{\sigma^2}-1}, & 0 \leq x \leq 1, \\ 1, & x > 1. \end{cases}$$

Definition

A continuous-time Markov process (X_t) with state space $(S, \mathcal{S})$ is said to be Harris recurrent if for some σ-finite measure ρ

$$P_x\Big(\int_0^\infty \mathbf{1}_{\{X_s \in A\}} ds = \infty\Big) = 1,$$

for any $x \in S$ and $A \in \mathcal{S}$ with $\rho(A) > 0$.

Definition

A Markov process (X_t) is said to be uniformly transient if

$$\sup_x E_x\Big[\int_0^\infty \mathbf{1}_K(X_t)dt\Big] < \infty$$

for every compact $K \subset S$.

Theorem (Kallenberg book Th. 20.17)

A regular Feller Process is either Harris Recurrent or Uniformly transient

Theorem (Jin, Mandrekar, Rüdiger , T.)

The CIR process $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}_x, x \in \mathbb{R}_+)$ *is Harris recurrent.*

Sketch of the proof

Lemma

The CIR process $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}_x, x \in \mathbb{R}_+)$ *is not uniformly transient.*

- Harris recurrence + regular Feller property guarantees the existence of a unique (up to multiplication by a constant) invariant measure for the Markov process.
- If this invariant measure is finite, then the process is called positive Harris recurrent.
- The CIR process has a finite invariant measure μ. It was shown in the original paper by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross (1985) that μ is a Gamma distribution :

$$\mu(dy) := \frac{\omega^{\nu}}{\Gamma(\nu)} y^{\nu-1} e^{-\omega y} dy, \quad y \geq 0,$$

where $\omega \equiv \frac{2a}{\sigma^2}$ and $\nu \equiv \frac{2a\theta}{\sigma^2}$.
$\Rightarrow$ The CIR process is positive Harris recurrent.

Corollary Let $X = (X_t)_{t\geq 0}$ be the CIR process. Then for any $f \in \mathcal{B}_b(\mathbb{R}_+)$ we have

$$\frac{1}{t}\int_0^t f(X_s)ds \;\rightarrow\; \int_{\mathbb{R}_+} f(x)\pi(dx) \quad a.s.$$

as $t \to \infty$, where π is the unique invariant probability measure of the CIR.

Proof.

The above convergence follows from Harris recurrence and [Theorem 20.21 book of Kallenberg]. □

Definition:

- The tail σ-field on $\hat{\Omega}$ is defined as $\hat{\mathcal{T}} := \cap_{t\geq 0}\sigma\{\hat{X}_s : s \geq t\}$.
- A σ-field $\mathcal{G} \subset \hat{\mathcal{F}}$ on $\hat{\Omega}$ is said to be $\hat{P}_\mu$-trivial if $\hat{P}_\mu(A) = 0$ or $\hat{P}_\mu(A) = 1$ for every $A \in \mathcal{G}$, where

$$\hat{P}_\mu(\cdot) := \int_{\mathbb{R}_+} \hat{P}_x(\cdot)\mu(dx)$$

From the Harris recurrence of the CIR process we reproduce the following well-known fact.

Corollary

The CIR process $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}_x, x \in \mathbb{R}_+)$ is strongly ergodic, meaning that the tail σ-field $\hat{\mathcal{T}}$ of the CIR process is $\hat{P}_\mu$-trivial for every μ.

Theorem (Jin, Mandrekar, Rüdiger , T.)

Suppose that $g : \mathbb{R}_+ \to \mathbb{R}$ *is continuous and* $f : \mathbb{R} \to \mathbb{R}_+$ *is measurable. Then for any* $x \in \mathbb{R}_+$ *we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f\Big(\int_j^{j+1} g(\hat{X}_s)(\hat{\omega})ds \Big) = \hat{E}_\mu \Big[f\Big(\int_0^1 g(\hat{X}_s)(\hat{\omega})ds \Big) \Big]$$

for $\hat{P}_x$*-almost all* $\hat{\omega} \in \hat{\Omega}$*, where* μ *is the unique invariant probability measure for the CIR process.*

The Proof uses positive Harris recurrence and other results like Birkhoff's Theorem in Kallenberg's book.

Introduction to some Application of the insurance company DeBeKa

Calibration of a model describing the credit migration model of Hurd and Kuznetsov. Consider the finite state space $\{1, 2, \cdots, 8\}$, which can be identified with Moody's rating classes via the mapping:

$$\{1, 2, \cdots, 8\} \leftrightarrow \{\text{AAA, AA, A}, \cdots, \text{default}\}.$$

The credit migration matrix $P(s, t), 0 \leq s \leq t$, is a stochastic 8×8 matrix and describes all possible transition probabilities between rating classes from time s to time t, namely

$$P(s, t) = \Big(p_{ij}(s, t)\Big)_{1 \leq i, j \leq 8}$$

where $p_{ij}(s, t)$ represents the transition probability from state i to state j from time s to time t.

Application of the insurance company DeBeKa

If we choose the functions of the previous Theorem in the following forms:

$$f(x) = e^{-x} \text{ and } g(x) = \hat{D}_{ii}x, \; 1 \leq i \leq 8,$$

where $\hat{D}_{ii}$ represent the eigenvalues of the generator matrix $\hat{P}$ in the credit migration model considered by Debeka.

We get the convergence results developed by B. Koehler and V. Krafft (2011), not only in L^2 but holds also in L^p for any $p \geq 1$.

Particular case of above JCIR: The Basic affine jump diffusion process (BAJD)

A particular JCIR is given by the BAJD -process, introduced by Duffie and Garleanu (2001) and analyzed also by Filipov (2001) and Keller -Ressel and Steiner (2008).

$$dX_t = a(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t + dJ_t, \quad X_0 = x \geq 0, \qquad (4)$$

where a, θ, σ are positive constants, W_t is a 1-dimensional Brownian motion and J_t is a pure-jump Lévy process with the Lévy measure

$$\nu(dy) = \begin{cases} cde^{-dy}dy, & y \geq 0, \\ 0, & y < 0, \end{cases}$$

for some constants $c \in \mathbb{R}_+$ and $d > 0$. The jump size and inter-arrival times are exponentially distributed with parameter d and c.

$$E_x\left[e^{uX_t}\right] = \exp\left(\phi(t,u) + x\psi(t,u)\right), \quad u \in \mathcal{U} := \{u \in \mathbb{C} : \Re u \le 0\},$$

where the functions $\phi(t,u)$ and $\psi(t,u)$ solve

$$\begin{cases} \partial_t \phi(t,u) = F\left(\psi(t,u)\right), & \phi(0,u) = 0, \\ \partial_t \psi(t,u) = R\left(\psi(t,u)\right), & \psi(0,u) = u, \end{cases}$$

$$F(u) = a\theta u + \frac{cu}{d-u}, \quad u \in \mathbb{C} \setminus \{d\},$$

$$R(u) = \frac{\sigma^2 u^2}{2} - au, \quad u \in \mathbb{C}.$$

By solving the system one can get $\phi(t,u)$, $\psi(t,u)$ and finally that the characteristic function of BAJD process X_t is

$$E_x[e^{uX_t}] =$$

$$\begin{cases} \left(1-\frac{\sigma^2}{2a}u(1-e^{-at})\right)^{-\frac{2a\theta}{\sigma^2}} \cdot \left(\frac{d-\frac{\sigma^2 du}{2a}+\left(\frac{\sigma^2 d}{2a}-1\right)ue^{-at}}{d-u}\right)^{\frac{c}{a-\frac{\sigma^2 d}{2}}} \\ \quad \cdot \exp\left(\frac{xue^{-at}}{1-\frac{\sigma^2}{2a}u(1-e^{-at})}\right), & \text{if } d \neq d_0, \\ \left(1-\frac{\sigma^2}{2a}u(1-e^{-at})\right)^{-\frac{2a\theta}{\sigma^2}} \exp\left(\frac{cu(1-e^{-at})}{a(d-u)}\right) \exp\left(\frac{xue^{-at}}{1-\frac{\sigma^2}{2a}u(1-e^{-at})}\right), & \text{if } d = d_0 \end{cases}$$

where $d_0 = \dfrac{2a}{\sigma^2}$.

Remember: if $f(t,x,y)$ is the density of CIR -process, then

$$\int_{\mathbb{R}_+} f(t,x,y)e^{uy}\,dy = \left(1-\frac{\sigma^2}{2a}u(1-e^{-at})\right)^{-\frac{2a\theta}{\sigma^2}} \cdot \exp\left(\frac{xue^{-at}}{1-\frac{\sigma^2}{2a}u(1-e^{-at})}\right).$$

We find an explicit formula for the density $p(t,x,y)$ of the BAJD -process, which in particular has the following form: in all cases we get

$$p(t,x,y) = L(t)f(t,x,y) + \int_0^y f(t,x,y-z)h(t,z)dz, \qquad (5)$$

where $L(t)$ is continuous in $t > 0$, $0 < L(t) < 1$ and $h(t,z)$ is non-negative, continuous in $(t,z) \in (0,\infty) \times [0,\infty)$ and $\int_R h(t,z)dz = 1 - L(t)$.
Moreover we prove **Theorem** [Jin, Rüdiger, T.]

Theorem

The BAJD- process $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}_x, x \in \mathbb{R}_+)$ *is positive Harris recurrent.*

Jump Diffusion CIR model (JCIR)

$$dX_t = a(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t + dJ_t, \quad X_0 = x \geq 0,$$

where $a, \sigma > 0$, $\theta \geq 0$, W_t is a 1-dimensional Brownian motion and J_t is a stoch. independent pure-jump Lévy process with its Lévy measure ν concentrating on $(0, \infty)$ and satisfying $\int_{(0,\infty)}(\xi \wedge 1)\nu(d\xi) < \infty$.

$$\begin{cases} \partial_t\phi(t,u) = F\big(\psi(t,u)\big), & \phi(0,u) = 0, \\ \partial_t\psi(t,u) = R\big(\psi(t,u)\big), & \psi(0,u) = u, \end{cases}$$

and

$$F(u) = a\theta u + \int_{(0,\infty)} (e^{u\xi} - 1)\nu(d\xi),$$

$$R(u) = \frac{\sigma^2 u^2}{2} - au.$$

By solving the ODEs one can get

$$\phi(t,u) = -\frac{2a\theta}{\sigma^2}\log\left(1-\frac{\sigma^2}{2a}u(1-e^{-at})\right) + \int_0^t \int_{(0,\infty)} \left(e^{\xi\psi(s,u)} - 1\right)\nu(d\xi)ds$$

and

$$\psi(t,u) = \frac{ue^{-at}}{1-\frac{\sigma^2}{2a}u(1-e^{-at})}$$

Characteristic function of JCIR

$$dX_t = a(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t + dJ_t, \quad X_0 = x \geq 0,$$

where $a, \sigma > 0$, $\theta \geq 0$

$$E_x[e^{uX_t}] = \left(1 - \frac{\sigma^2}{2a}u(1 - e^{-at})\right)^{-\frac{2a\theta}{\sigma^2}} \cdot \exp\left(\frac{xue^{-at}}{1 - \frac{\sigma^2}{2a}u(1 - e^{-at})}\right)$$
$$\cdot \exp\left(\int_0^t \int_0^\infty \left(e^{\xi\psi(s,u)} - 1\right)\nu(d\xi)ds\right).$$

Set

$$I := \left(1 - \frac{\sigma^2}{2a}u(1 - e^{-at})\right)^{-\frac{2a\theta}{\sigma^2}} \cdot \exp\left(\frac{xue^{-at}}{1 - \frac{\sigma^2}{2a}u(1 - e^{-at})}\right)$$
$$II := \exp\left(\int_0^t \int_0^\infty \left(e^{\xi\psi(s,u)} - 1\right)\nu(d\xi)ds\right).$$

Special case 1: ν=0, no jumps!

CIR

$$dY_t = a(\theta - Y_t)dt + \sigma\sqrt{Y_t}dW_t, \quad Y_0 = x \geq 0$$

$$E_x[e^{uY_t}] = \left(1 - \frac{\sigma^2}{2a}u(1-e^{-at})\right)^{-\frac{2a\theta}{\sigma^2}} \cdot \exp\left(\frac{xue^{-at}}{1-\frac{\sigma^2}{2a}u(1-e^{-at})}\right)$$
$$= I$$

- Y_t has a density function $f(t,x,y)$

Special case 2: θ=0 and $x = 0$

JCIR (θ=0 and $x = 0$):

$$dZ_t = -aZ_t dt + \sigma\sqrt{Z_t}dW_t + dJ_t, \quad Z_0 = 0 \geq 0$$

$$E_x[e^{uZ_t}] = \exp\left(\int_0^t \int_{(0,\infty)} \left(e^{\xi\psi(s,u)} - 1\right)\nu(d\xi)ds\right) = II$$

where

$$\psi(t,u) = \frac{ue^{-at}}{1 - \frac{\sigma^2}{2a}u(1 - e^{-at})}$$

- II resembles the characteristic function of a compound Poisson distribution

Remember that

$$II = E_x[e^{uZ_t}] = \exp\left(\int_0^t \int_{(0,\infty)} \left(e^{\xi\psi(s,u)} - 1\right)\nu(d\xi)ds\right)$$

where

$$\psi(t,u) = \frac{ue^{-at}}{1 - \frac{\sigma^2}{2a}u(1 - e^{-at})}$$

Theorem 1

Suppose that $\int_{(0,1)} \xi \ln(\frac{1}{\xi})\nu(d\xi) < \infty$. Then II is the characteristic function of a compound Poisson distribution. In particular, $\mathbb{P}_x(Z_t = 0) > 0$.

Theorem 2 (Lower bound for the transition density of JCIR)

Suppose that $\int_{(0,1)} \xi \ln(\frac{1}{\xi})\nu(d\xi) < \infty$. Then for all $A \in \mathcal{B}(\mathbb{R}_+)$,

$$\mathbb{P}(X_t \in A) \geq C(t) \int_A f(t,x,y)dy,$$

where $C(t) > 0$ and $f(t,x,y)$ is the transition density of the CIR process without jumps.

Corollary: if $\int_{(0,1)} \xi \ln(\frac{1}{\xi})\nu(d\xi) < \infty$, then the JCIR process X_t is irreducible

Suppose that

$$\int_{(1,\infty)} \xi \, \nu(d\xi) < \infty \quad \text{and} \quad \int_{(0,1)} \xi \ln(\frac{1}{\xi}) \nu(d\xi) < \infty.$$

Theorem

[M. Keller Ressel] JCIR has a unique invariant prob. measure π.

Theorem

The JCIR process X_t is exponential ergodic, namely there exist constants $0 < \beta < 1$ and $0 < B < \infty$ such that

$$\|P_t(x,\cdot) - \pi\|_{TV} \leq B\big(x+1\big)\beta^t, \quad t \geq 0, \quad x \in \mathbb{R}_+.$$

where $\|\cdot\|_{TV}$ denotes the total-variation norm for signed measures on $\mathbf{R}_+$, namely

$$\|\mu\|_{TV} = \sup_{A \in \mathcal{B}(\mathbf{R}_+)} \{|\mu(A)|\}.$$

Proof: For any $\delta > 0$ we consider the δ-skeleton chain $Y_n^\delta := X_{n\delta},\ n \in \mathbb{Z}_+$. Then $(Y_n^\delta)_{n\in\mathbb{Z}_+}$ is a Markov chain with transition kernel $p(\delta, x, y)$ on the state space $\mathbb{R}_+$ with same invariant measure π.

- The CIR process Y_t is irreducible, aperiodic.
- It follows as Corollary that he JCIR process X_t is irreducible, aperiodic. (This is shown using Theorem 2)
- JCIR is a Feller Process
- There exists a Foster -Lyapunov function.

$$dX_t = a(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t + dJ_t, \quad X_0 = x \geq 0,$$

Theorem 3 (Existence of a Lyapunov function for JCIR)

Suppose that $\int_{(1,\infty)} \xi \, \nu(d\xi) < \infty$. Then the function $V(x) = x, \; x \geq 0$ is a Lyapunov funciton for the JCIR process X_t, i.e. for all $t > 0, \; x \geq 0$,

$$E_x[V(X_t)] \leq e^{-at}V(x) + M,$$

where $0 < M < \infty$ is a constant.

Summary

To prove strong ergodicity properties and analyse their speed of convergence (exp. ergodicity properties) we used

- affine properties of JCIR
- that JCIR can be obtained by convolution with CIR

Actually, I am trying to skip either one or the other property for some further interest rate model with jumps used by insurance companies.

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