Ill-posedness of stochastic incompressible Euler equations with passive tracer driven by a Stratonovich noise

International Conference on Stochastic Calculus and Applications to Finance

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June 05, 2024

Outline of presentation

Introduction

For $n \geq 2$, we consider the following **stochastic incompressible Euler equations with passive tracer** defined on $\mathbb{R}^n \times [0, \infty)$:

$$\mathrm{d}\zeta(t) + \nabla\zeta(t) \cdot \mathbf{z}(t)\mathrm{d}t = 0, \tag{PTE}$$

$$\mathrm{d}\mathbf{z}(t) + \mathrm{div}\,(\mathbf{z}(t) \otimes \mathbf{z}(t))\mathrm{d}t + \nabla\pi(t)\,\mathrm{d}t = -\gamma\mathbf{z}(t) \circ \mathrm{d}B(t), \tag{SEE}$$

$$\mathrm{div}\,\mathbf{z}(t) = 0, \tag{INE}$$

where $\mathbf{z}(t) = \mathbf{z}(x,t) \in \mathbb{R}^n$ is the velocity, $\zeta(t) = \zeta(x,t) \in \mathbb{R}$ is the tracer, $\pi(t) = \pi(x,t) \in \mathbb{R}$ is the pressure, $\gamma > 0$ is a constant, $\circ$ means that the stochastic integral is understood in the sense of Stratonovich and $B(\cdot)$ is a real-valued Brownian motion.

Motivation

- The idea of considering the incompressible Euler equations with passive tracer to any **dimension $\geq$ 2** is motivated by the work *Bronzi, et al.*[2], in which the authors proved the nonuniqueness of **2D incompressible ideal flow with passive tracer**.

- To generate the **infinitely many weak solution** we adopt the Baire-category method[1], instead of *convex integration technique*, which was left by *Bronzi, et al.*[2].

- Considering a linear multiplicative Stratonovich forcing to the incompressible Euler equations with passive tracer came from the work of *Chiodaroli, et al.*[3], where the authors proved the **existence of infinitely many global-in-time weak solutions to compressible Euler equations driven by multiplicative white noise**.

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[1] C.D. Lellis and L. Székelyhidi, The Euler equations as a differential inclusion, *Ann. Math.*, **170**, 1417–1436, 2009.

[2] A.C. Bronzi, M.C.F. Filho and H.J.N. Lopes, Wild solutions for 2D incompressible ideal flow with passive tracer, *Commun. Math. Sci.*, **13**, 1333–1343, 2015.

[3] E. Chiodaroli, E. Feireisl and F. Flandoli, Ill-posedness for the full Euler system driven by multiplicative white in time noise, *Indiana Univ. Math. J.*, **70**, 1267–1282, 2021.

Weak solution

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. We say that a vector

$$(\zeta, \mathbf{z}) \in \mathrm{C}_{\text{weak}}(\mathbb{R}^n \times [0,\infty); \mathbb{R} \times \mathbb{R}^n) \cap \mathrm{L}^{\infty}_{\text{loc}}(\mathbb{R}^n \times [0,\infty); \mathbb{R} \times \mathbb{R}^n)$$

is a *weak solution* (in the **analytic sense**) of the problem (PTE)-(INE) with the initial condition $\zeta(0) = \zeta_0$ and $\mathbf{z}(0) = \mathbf{z}_0$, if the following hold:
The functions

$$t \mapsto \int_{\mathbb{R}^n} \zeta(t)\varphi(x)\mathrm{d}x,\ t \mapsto \int_{\mathbb{R}^n} \mathbf{z}(t) \cdot \boldsymbol{\varphi}(x)\mathrm{d}x,$$

are continuous $\{\mathcal{F}_t\}_{t\geq 0}$-adapted **semimartingales** $\mathbb{P}$-a.s. for any scalar field $\varphi \in \mathrm{C}^1(\mathbb{R}^n; \mathbb{R})$ and vector field $\boldsymbol{\varphi} \in \mathrm{C}^1(\mathbb{R}^n; \mathbb{R}^n)$, respectively, and for any $\tau \geq 0$

$$\int_{\mathbb{R}^n} \zeta(\tau)\varphi(x)\mathrm{d}x = \int_{\mathbb{R}^n} \zeta_0\varphi(x)\mathrm{d}x + \int_0^\tau \int_{\mathbb{R}^n} (\zeta(t)\mathbf{z}(t)) \cdot \nabla(x)\varphi(x)\mathrm{d}x\mathrm{d}t,$$

$$\int_{\mathbb{R}^n} \mathbf{z}(\tau) \cdot \boldsymbol{\varphi}(x)\mathrm{d}x = \int_0^\tau \int_{\mathbb{R}^n} ((\mathbf{z}(t) \otimes \mathbf{z}(t)) : \nabla\boldsymbol{\varphi}(x) + \pi(t)\,\mathrm{div}\,\boldsymbol{\varphi}(x))\mathrm{d}x\mathrm{d}t$$
$$+ \int_{\mathbb{R}^n} \mathbf{z}_0 \cdot \boldsymbol{\varphi}(x)\mathrm{d}x - \gamma \int_0^\tau \left(\int_{\mathbb{R}^n} \mathbf{z}(x) \cdot \boldsymbol{\varphi}(x)\mathrm{d}x \right) \circ \mathrm{d}B(t).$$

Non-uniqueness result for SPDEs

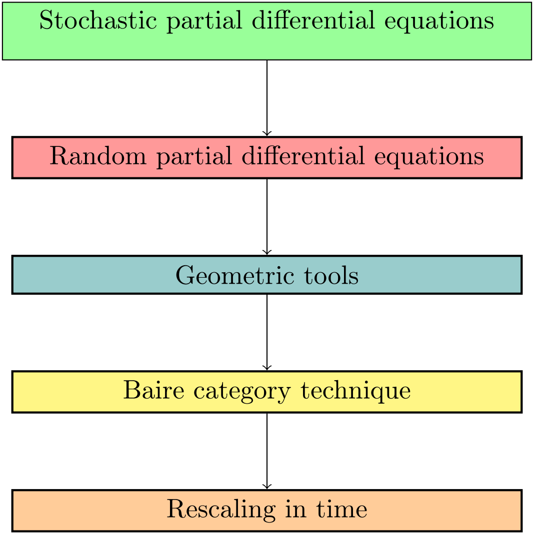
Let $\mathcal{O} \subset \mathbb{R}^n \times [0,\infty)$ be a **bounded domain**. Then there exist infinitely many weak solutions $(\zeta, \mathbf{z})$ of the problem (PTE)-(INE) in the sense of previous definition such that

(i) $|\mathbf{z}(\theta(t))| = \theta'(t)$ and $|\zeta(\theta(t))| = 1$ a.e. in $\mathcal{O}$;

(ii) $\mathbf{z}(\theta(t)) = \mathbf{0}$, $\zeta(\theta(t)) = 0$ and $\pi(\theta(t)) = 0$ a.e. in $\mathbb{R}^n \times [0,\infty)\backslash\mathcal{O}$,

where $\theta(t) = \int_0^t e^{-\gamma B(s)} \mathrm{d}s$.

Plan of this talk



Approaching to random PDEs

Using the transformation $\boldsymbol{v}(t) = \phi(t)\mathbf{z}(t)$, where $\phi(t) = e^{\gamma B(t)}$, the SPDEs (SEE) transforms in the following random PDEs:

$$\frac{\partial \boldsymbol{v}(t)}{\partial t} = -e^{-\gamma B(t)}\mathrm{div}\ (\boldsymbol{v}(t) \otimes \boldsymbol{v}(t)) - e^{\gamma B(t)}\nabla \pi(t). \tag{1}$$

Let us consider the time transformation[3]

$$t \mapsto \int_0^t e^{-\gamma B(s)}\, \mathrm{d}s =: \theta(t).$$

Thus, equation (1) becomes

$$e^{-\gamma B(t)}\left[\frac{\mathrm{d}\boldsymbol{v}(\theta)}{\mathrm{d}\theta} + \mathrm{div}\ (\boldsymbol{v}(\theta) \otimes \boldsymbol{v}(\theta)) + e^{2\gamma B(t)}\nabla \pi(\theta)\right] = \mathbf{0}.$$

Since $e^{-\gamma B(t)} \neq 0$, for $p(\theta) = e^{2\gamma B(t)}\pi(\theta) = \pi(\theta)(\frac{\mathrm{d}\theta}{\mathrm{d}t})^{-2}$, it is immediate that

$$\frac{\mathrm{d}\boldsymbol{v}(\theta)}{\mathrm{d}\theta} + \mathrm{div}\ (\boldsymbol{v}(\theta) \otimes \boldsymbol{v}(\theta)) + \nabla p(\theta) = \mathbf{0}. \tag{REE}$$

[3] E. Chiodaroli, E. Feireisl and F. Flandoli, Ill-posedness for the full Euler system driven by multiplicative white noise, *Indiana Univ. Math. J.*, **70**, 1267–1282, 2021.

Random PDEs

Similarly, with the help of the following substitution:

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \frac{\mathrm{d}\zeta}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = e^{-\gamma B(t)}\frac{\mathrm{d}\zeta}{\mathrm{d}\theta},$$

we obtain the passive tracer equation (PTE) as

$$e^{-\gamma B(t)}\left[\frac{\mathrm{d}\zeta}{\mathrm{d}\theta} + \nabla\zeta\cdot(e^{-\gamma B(t)}\mathbf{z})\right] = \frac{\mathrm{d}\zeta}{\mathrm{d}\theta} + \nabla\zeta\cdot\boldsymbol{v} = 0.$$

By clubbing above equation with (REE) and (INE), we arrive at the random incompressible Euler system with passive tracer.

$$\begin{cases} \zeta_\theta(\theta) + \operatorname{div}(\zeta(\theta)\boldsymbol{v}(\theta)) = 0, \\ \boldsymbol{v}_\theta(\theta) + \operatorname{div}(\boldsymbol{v}(\theta)\otimes\boldsymbol{v}(\theta)) + \nabla p(\theta) = \mathbf{0}, \\ \operatorname{div}\boldsymbol{v}(\theta) = 0. \end{cases} \tag{RIEEPT}$$

Weak solution of random PDEs

A pair $(\zeta, \boldsymbol{v}) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^n \times [0,\infty); \mathbb{R} \times \mathbb{R}^n)$, is called a *weak solution* of the random incompressible Euler equations with passive tracer (RIEEPT) if for a.e. $\theta \in [0,\infty)$ and for any test functions $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^n \times (0,\infty); \mathbb{R})$ and $\boldsymbol{\varphi} \in \mathrm{C}_0^\infty(\mathbb{R}^n \times (0,\infty); \mathbb{R}^n)$ with $\operatorname{div} \boldsymbol{\varphi}(\theta) = 0$, $(\zeta, \boldsymbol{v})$ satisfies

$$\int_0^\infty \int_{\mathbb{R}^n} \zeta(\theta) \cdot \varphi_\theta(\theta)\, \mathrm{d}x \mathrm{d}\theta + \int_0^\infty \int_{\mathbb{R}^n} (\zeta(\theta)\boldsymbol{v}(\theta)) \cdot \nabla \varphi(\theta)\, \mathrm{d}x \mathrm{d}\theta = 0,$$

$$\int_0^\infty \int_{\mathbb{R}^n} \boldsymbol{v}(\theta) \cdot \boldsymbol{\varphi}_\theta(\theta)\, \mathrm{d}x \mathrm{d}\theta + \int_0^\infty \int_{\mathbb{R}^n} (\boldsymbol{v}(\theta) \otimes \boldsymbol{v}(\theta)) : \nabla \boldsymbol{\varphi}(\theta)\, \mathrm{d}x \mathrm{d}\theta = 0,$$

$$\int_0^\infty \int_{\mathbb{R}^n} \boldsymbol{v}(\theta) \cdot \nabla \boldsymbol{\varphi}(\theta)\, \mathrm{d}x \mathrm{d}\theta = 0.$$

Non-uniqueness result for random PDEs

Let $\mathcal{O} \subset \mathbb{R}^n \times [0, \infty)$ be a bounded domain. Then there exist **infinitely many weak solutions**

$$(\zeta, \boldsymbol{v}) \in \mathrm{C}_{\text{weak}}(\mathbb{R}^n \times [0,\infty); \mathbb{R} \times \mathbb{R}^n) \cap \mathrm{L}^{\infty}_{\text{loc}}(\mathbb{R}^n \times [0,\infty); \mathbb{R} \times \mathbb{R}^n)$$

of the problem (RIEEPT) such that

(i) $|\boldsymbol{v}(\theta)| = 1$ and $|\zeta(\theta)| = 1$ a.e. in $\mathcal{O}$;

(ii) $\boldsymbol{v}(\theta) = \mathbf{0}$, $\zeta(\theta) = 0$ and $p(\theta) = 0$ a.e. in $\mathbb{R}^n \times [0, \infty)\backslash\mathcal{O}$.

Geometric setup

The Euler equations with passive tracer can be naturally rewritten in the Tartar's framework[4]. Thus, we can rewrite the (RIEEPT) as the following differential equations:

$$\begin{cases} \zeta_\theta + \operatorname{div} \boldsymbol{\eta} = 0, \\ \boldsymbol{v}_\theta + \operatorname{div} \boldsymbol{u} + \nabla q = \mathbf{0}, \\ \operatorname{div} \boldsymbol{v} = 0, \end{cases} \tag{TF}$$

with

$$q = p + \frac{|\boldsymbol{v}|^2}{n}, \ \boldsymbol{u} = \boldsymbol{v} \otimes \boldsymbol{v} - \frac{|\boldsymbol{v}|^2}{n} I_n, \quad \text{and } \boldsymbol{\eta} = \zeta \boldsymbol{v},$$

where $\boldsymbol{u}(\cdot,\cdot)$ belongs to the space of trace-free $n \times n$ real symmetric matrices, denoted by $\mathbb{S}_0^n$.

[4] L. Tartar, Compensated compactness and applications to partial differential equations, *Res. Notes in Math.*, **36**, 136–212, 1979.

Constraint set

We define the graph or constraint set $\mathcal{K}$ by $\mathcal{K} := K \times [-1,1]$ with

$$K := \left\{ (\widetilde{\zeta}, \widetilde{\boldsymbol{\eta}}, \widetilde{\boldsymbol{v}}, \widetilde{\boldsymbol{u}}) \in \{-1,1\} \times \mathcal{S}^{n-1} \times \mathcal{S}^{n-1} \times \mathbb{S}_0^n : \widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{v}} \otimes \widetilde{\boldsymbol{v}} - \frac{|\widetilde{\boldsymbol{v}}|^2}{n} I_n \text{ and } \widetilde{\boldsymbol{\eta}} = \widetilde{\zeta}\widetilde{\boldsymbol{v}} \right\}$$

and

$$\mathcal{U} = \operatorname{int}(K^{\mathrm{co}} \times [-1,1]),$$

where $\mathcal{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$, K^{co} represents the **convex hull** of K, and int stands for the **topological interior** of the set in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}_0^n \times \mathbb{R}$.

Remark

(i) One can adapt similar arguments in the proof of Lemma 4.2[1] *to show that* $\mathbf{0} \in \mathcal{U}$*. Thus,* $\mathcal{U}$ *is non-empty.*
(ii) Any solution $(\zeta, \boldsymbol{\eta}, \boldsymbol{v}, \boldsymbol{u}, q)$ *of* (TF) *with image contained in* $\mathcal{K}$ *is a solution of* (RIEEPT).

[1] C.D. Lellis and L. Székelyhidi, The Euler equations as a differential inclusion, *Ann. Math.*, **170**, 1417–1436, 2009.

Matrix form

Let us introduce the following $(n+2)\times(n+1)$ matrix field

$$U = \begin{pmatrix} \boldsymbol{u} + qI_n & \boldsymbol{v} \\ \boldsymbol{v}^\top & 0 \\ \boldsymbol{\eta}^\top & \zeta \end{pmatrix} \tag{2}$$

and a new coordinate system $y = (x, \theta) \in \mathbb{R}^n \times [0, \infty)$, where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. In this setting, the equation (TF) reduces to

$$\operatorname{div}_y U = \mathbf{0}. \tag{EMF}$$

Let $\mathcal{M}_{(n+1)\times(n+1)}$ be the set of symmetric $(n+1)\times(n+1)$ matrices A such that $A_{n+1,n+1} = 0$ and suppose the set of $(n+2)\times(n+1)$ matrices A such that $(A_{i,j})_{1\le i,j\le n+1} \in \mathcal{M}_{(n+1)\times(n+1)}$. Observe that the following linear map is isomorphism

$$\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}_0^n\times\mathbb{R} \ni (\zeta, \boldsymbol{\eta}, \boldsymbol{v}, \boldsymbol{u}, q) \mapsto \begin{pmatrix} \boldsymbol{u} + qI_n & \boldsymbol{v} \\ \boldsymbol{v}^\top & 0 \\ \boldsymbol{\eta}^\top & \zeta \end{pmatrix} \in \mathcal{M}_{(n+2)\times(n+1)}.$$

Plane wave solution

A plane wave solution of (EMF) is a solution U, as in (2), of the form

$$U = U(y) = Ah(y \cdot \xi),$$

where $h : \mathbb{R} \to \mathbb{R}$ and $A \in \mathcal{M}_{(n+2)\times(n+1)}$.
Then the **wave cone** is the set of states of the planar solution of (EMF) for any h.
In our case, the wave cone is given by

$$\Lambda = \{A \in \mathcal{M}_{(n+2)\times(n+1)} : \text{ there exists } \xi \in \mathbb{R}^{n+1} \backslash \{\mathbf{0}\} \text{ such that } A\xi = \mathbf{0}\},$$

or, equivalently,

$$\Lambda = \Big\{ (\widetilde{\zeta}, \widetilde{\boldsymbol{\eta}}, \widetilde{\boldsymbol{v}}, \widetilde{\boldsymbol{u}}, \widetilde{q}) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^n_0 \times \mathbb{R}$$
$$: \text{there exists } \xi \in \mathbb{R}^{n+1} \backslash \{\mathbf{0}\} \text{ such that } \begin{pmatrix} \widetilde{\boldsymbol{u}} + \widetilde{q} I_n & \widetilde{\boldsymbol{v}} \\ \widetilde{\boldsymbol{v}}^\top & 0 \\ \widetilde{\boldsymbol{\eta}}^\top & \widetilde{\zeta} \end{pmatrix} \xi = \mathbf{0} \Big\}.$$

Geometric lemma

Lemma

There exists a dimensional constant $C > 0$ *such that for each* $(\zeta, \boldsymbol{\eta}, \boldsymbol{v}, \boldsymbol{u}, q) \in \mathcal{U}$*, there exists* $(\bar{\zeta}, \bar{\boldsymbol{\eta}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{u}}) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^n_0$ *satisfying*

(i) $(\bar{\zeta}, \bar{\boldsymbol{\eta}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{u}}, 0) \in \Lambda$;

(ii) *the line segment with endpoints* $(\zeta, \boldsymbol{\eta}, \boldsymbol{v}, \boldsymbol{u}, q) \pm (\bar{\zeta}, \bar{\boldsymbol{\eta}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{u}}, 0) \in \mathcal{U}$;

(iii) $\left|(\bar{\boldsymbol{v}}, \bar{\zeta})\right| \geq C(2 - (|\boldsymbol{v}|^2 + |\zeta|^2))$.

Localized plane waves

Proposition

Let $\widehat{V} \in \Lambda$ be such that $\widehat{V} e_{n+1} \neq \mathbf{0}$, and consider the line segment σ with endpoints $-\widehat{V}$ and $\widehat{V}$ in $\mathcal{M}_{(n+2)\times(n+1)}$. Then there exists a constant $\alpha > 0$ such that for any $\varepsilon > 0$ there exists a smooth divergence-free *matrix field*

$$V : \mathbb{R}^n \times [0,\infty) \to \mathcal{M}_{(n+2)\times(n+1)} \text{ given by } V(y) = \begin{pmatrix} \boldsymbol{u}(y) + q(y) I_n & \boldsymbol{v}(y) \\ \boldsymbol{v}^\top(y) & 0 \\ \boldsymbol{\eta}^\top(y) & \zeta(y) \end{pmatrix}$$

where $\boldsymbol{u}(y) \in \mathbb{S}^n_0$, $\boldsymbol{v}(y), \boldsymbol{\eta}(y) \in \mathbb{R}^n$ and $\zeta(y), q(y) \in \mathbb{R}$, with the properties

(p1) $\operatorname{supp}(V) \subset B_1(\mathbf{0})$;

(p2) $\operatorname{dist}(V(y), \sigma) < \varepsilon$ *for all* $y \in B_1(\mathbf{0})$;

(p3) $\displaystyle\int_{B_1(\mathbf{0})} |\boldsymbol{v}(y)| \mathrm{d}y \geq \alpha |\widehat{\boldsymbol{v}}|$ *and* $\displaystyle\int_{B_1(\mathbf{0})} |\zeta(y)| \mathrm{d}y \geq \alpha |\widehat{\zeta}|$;

where $\alpha > 0$ is a dimensional constant, $y = (x, \theta) \in \mathbb{R}^n \times [0,\infty)$ and $B_1(\mathbf{0})$ denotes the open ball of radius 1 centered at $\mathbf{0}$ in $\mathbb{R}^n \times [0,\infty)$.

Space of subsolutions

We define the complete metric space X with the help of the following *subsolutions space*:

Definition

Let X_0 denote the set of functions

$$(\zeta, \boldsymbol{\eta}, \boldsymbol{v}, \boldsymbol{u}, q) \in \mathrm{C}^\infty(\mathbb{R}^n \times [0, \infty); \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}_0^n \times \mathbb{R})$$

that satisfies

(i) $\operatorname{supp}(\zeta, \boldsymbol{\eta}, \boldsymbol{v}, \boldsymbol{u}, q) \subset \mathcal{O}$;

(ii) $(\zeta, \boldsymbol{\eta}, \boldsymbol{v}, \boldsymbol{u}, q)$ solves (TF) in $\mathbb{R}^n \times [0, \infty)$;

(iii) $(\zeta, \boldsymbol{\eta}, \boldsymbol{v}, \boldsymbol{u}, q)(y) \in \mathcal{U}$ for all $y \in \mathbb{R}^n \times [0, \infty)$.

- We equip X_0 with the L^∞-weak* convergence topology and then define X the closure of X_0 in this topology.
- Let us fix a metric d^*_∞ inducing the weak* topology of L^∞ in X, so that (X, d^*_∞) is a complete metric space

.

Metrizable space

Lemma

The set X *with* L^∞*-weak* convergence topology is a non-empty compact metrizable space. Moreover, if* $(\zeta, \boldsymbol{\eta}, \boldsymbol{v}, \boldsymbol{u}, q) \in \mathrm{X}$ *is such that*

$$|\boldsymbol{v}(y)| = 1 \text{ and } |\zeta(y)| = 1 \text{ for a.e. } y \in \mathcal{O},$$

then $\boldsymbol{v}$, ζ *and* $p := q - \frac{|\boldsymbol{v}|^2}{2}$ *is a weak solution of (RIEEPT) such that*

$$\boldsymbol{v}(y) = \boldsymbol{0},\ \zeta(y) = 0 \text{ and } p(y) = 0 \text{ for all } y \in \mathbb{R}^n \times \mathbb{R} \backslash \mathcal{O}.$$

Lemma

There exists a constant $\beta > 0$ *with the following properties: Given* $(\zeta_0, \boldsymbol{\eta}_0, \boldsymbol{v}_0, \boldsymbol{u}_0, q_0) \in \mathrm{X}_0$ *there exists a sequence* $(\zeta_k, \boldsymbol{\eta}_k, \boldsymbol{v}_k, \boldsymbol{u}_k, q_k) \in \mathrm{X}_0$ *such that*

$$\begin{aligned} &\|\boldsymbol{v}_k\|^2_{\mathrm{L}^2(\mathcal{O})} + \|\zeta_k\|^2_{\mathrm{L}^2(\mathcal{O})} \\ &\geq \|\boldsymbol{v}_0\|^2_{\mathrm{L}^2(\mathcal{O})} + \|\zeta_0\|^2_{\mathrm{L}^2(\mathcal{O})} + \beta\big(2|\mathcal{O}| - (\|\boldsymbol{v}_0\|^2_{\mathrm{L}^2(\mathcal{O})} + \|\zeta_0\|^2_{\mathrm{L}^2(\mathcal{O})})\big)^2, \end{aligned}$$

and

$$(\zeta_k, \boldsymbol{\eta}_k, \boldsymbol{v}_k, \boldsymbol{u}_k, q_k) \xrightarrow[k\to\infty]{*} (\zeta_0, \boldsymbol{\eta}_0, \boldsymbol{v}_0, \boldsymbol{u}_0, q_0) \text{ in } \mathrm{L}^\infty(\mathcal{O}).$$

Baire-1 map

Definition

In a metric space X, a function $J : \mathrm{X} \to \mathbb{R}$ is a *Baire-1 map*[5] if it is a **pointwise limit** of continuous functions.

Example

Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions defined by $f_n(x) = x^n$. Then, the pointwise limit of $\{f_n\}_{n\in\mathbb{N}}$ is $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0, & \text{if } x \in [0,1), \\ 1, & \text{if } x = 1, \end{cases} \quad \text{is a } \textit{Baire-1 map.}$$

Theorem

*If $J : \mathrm{X} \to \mathbb{R}$ is a Baire-1 map on a complete metric space X, then the **set of continuity points** of J is a dense[5] set in X.*

[5] L. Székelyhidi, From isometric embeddings to turbulence, HCDTE lecture notes. Part II. Nonlinear hyperbolic PDEs, dispersive and transport equations, *AIMS Ser. Appl. Math.*,**7**, 63, Am. Inst. Math. Sci. (AIMS), Springfield, MO, 2013.

Baire-category method

Lemma

The identity map

$$I : (\mathrm{X}, d^*_\infty) \to \mathrm{L}^2(\mathbb{R}^n \times [0,\infty); \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^n_0 \times \mathbb{R})$$

defined by

$$(\zeta, \boldsymbol{\eta}, \boldsymbol{v}, \boldsymbol{u}, q) \mapsto (\zeta, \boldsymbol{\eta}, \boldsymbol{v}, \boldsymbol{u}, q)$$

is a Baire-1 map and therefore the set of points of continuity is dense in (X, d^*_∞).

Lemma

If $(\zeta, \boldsymbol{\eta}, \boldsymbol{v}, \boldsymbol{u}, q) \in \mathrm{X}$ *is a point of continuity of* I*, then*

$$|\boldsymbol{v}(y)| = 1 \text{ and } |\zeta(y)| = 1 \text{ for a.e. } y \in \mathcal{O}.$$

How to go back?

`How to go back to the solutions z of the original stochastic system (PTE)-(INE) from the solution` $\boldsymbol{v}$, `constructed from Baire-category method, of random PDEs (RIEEPT)?`

Consider the following time function

$$[0,\infty) \ni \theta \mapsto \int_{\mathbb{R}^n} \boldsymbol{v}(\theta)\cdot\boldsymbol{\varphi}(x)\,\mathrm{d}x = \langle \boldsymbol{v},\boldsymbol{\varphi}\rangle,\ \boldsymbol{\varphi}\in \mathrm{C}_0^1(\mathbb{R}^n;\mathbb{R}^n).$$

Now, observe that $\langle \boldsymbol{v},\boldsymbol{\varphi}\rangle$ is a globally Lipchitz function in time, which is immediate from weak formulation of (REE). Then, its times derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{v},\boldsymbol{\varphi}\rangle = \partial_t \int_{\mathbb{R}^n} \boldsymbol{v}\Big(\int_0^t e^{-\gamma B(s)}ds\Big)\cdot\boldsymbol{\varphi}(x)\mathrm{d}x = e^{-\gamma B(t)}\int_{\mathbb{R}^n} \boldsymbol{v}_t(t)\cdot\boldsymbol{\varphi}(x)\mathrm{d}x.$$

By using the integration by parts formula, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{v},\boldsymbol{\varphi}\rangle = e^{-\gamma B(t)}\int_{\mathbb{R}^n}\left[\boldsymbol{v}(t)\otimes\boldsymbol{v}(t) : \nabla\boldsymbol{\varphi}(x) + p(t)\,\mathrm{div}\ \boldsymbol{\varphi}(x)\right]\mathrm{d}x. \tag{3}$$

Now, we are in position to define the velocity

$$\mathbf{z}(t) = e^{-\gamma B(t)}\boldsymbol{v}(t).$$

Rescaling time

Using the basic properties of Stratonovich integral, we obtain

$$\mathrm{d}(be^{B(t)}) = e^{B(t)}\mathrm{d}b + be^{B(t)} \circ \mathrm{d}B(t), \quad \text{provided } b \text{ is Lipschitz.} \tag{4}$$

Considering LHS of (3) and using (4), we obtain

$$\begin{aligned}\mathrm{d}\int_{\mathbb{R}^n} \boldsymbol{v}(t)\cdot\boldsymbol{\varphi}\mathrm{d}x &= \mathrm{d}\left[e^{\gamma B(t)}\int_{\mathbb{R}^n} \mathbf{z}(t)\cdot\boldsymbol{\varphi}(x)\mathrm{d}x\right]\\ &= e^{\gamma B(t)}\left[\mathrm{d}\int_{\mathbb{R}^n} \mathbf{z}(t)\cdot\boldsymbol{\varphi}(x)\mathrm{d}x + \gamma\int_{\mathbb{R}^n} \mathbf{z}(t)\cdot\boldsymbol{\varphi}(x)\mathrm{d}x \circ \mathrm{d}B(t)\right]\end{aligned} \tag{5}$$

Then, comparing the RHS of (3) and (5), we assert

$$\begin{aligned}&e^{\gamma B(t)}\left[\mathrm{d}\int_{\mathbb{R}^n} \mathbf{z}(t)\cdot\varphi(x)\mathrm{d}x + \gamma\int_{\mathbb{R}^n} \mathbf{z}(t)\cdot\varphi(x)\mathrm{d}x \circ \mathrm{d}B(t)\right]\\ &\qquad= e^{-\gamma B(t)}\left[\int_{\mathbb{R}^n} \boldsymbol{v}(t)\otimes\boldsymbol{v}(t) : \nabla\varphi(x)\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^n} p(t)\,\mathrm{div}\;\varphi(x)\,\mathrm{d}x\mathrm{d}t\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{d}\int_{\mathbb{R}^n} \mathbf{z}(t)\cdot\varphi(x)\mathrm{d}x &= \int_{\mathbb{R}^n} \mathbf{z}(t)\otimes\mathbf{z}(t) : \nabla\varphi(x)\mathrm{d}x\mathrm{d}t\\ &\quad+ \int_{\mathbb{R}^n} \pi(t)\,\mathrm{div}\;\varphi(x)\,\mathrm{d}x\mathrm{d}t - \gamma\int_{\mathbb{R}^n} \mathbf{z}(t)\cdot\varphi(x)\mathrm{d}x \circ \mathrm{d}B(t).\end{aligned}$$

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“Fluid dynamicists are divided into hydraulic engineers who observe what could not be explained, and mathematicians who explain things that cannot be observed.”

Cyril N. Hinshelwood 1956

Thank you for your attention!