Optimal Options Portfolio

Shashi Jain
Associate Professor
Department of Management Studies
Indian Institute of Science

Credits

- Started with Dr Samrat Sen (Department of Mathematics, IISc)
- Meghal S (BS/MS, IISc), Atharva Bhide (IISER)
- Abhiram M (BS project, IISc)
- Also thank Ankit Gupta and team from Fidelity for the discussions.

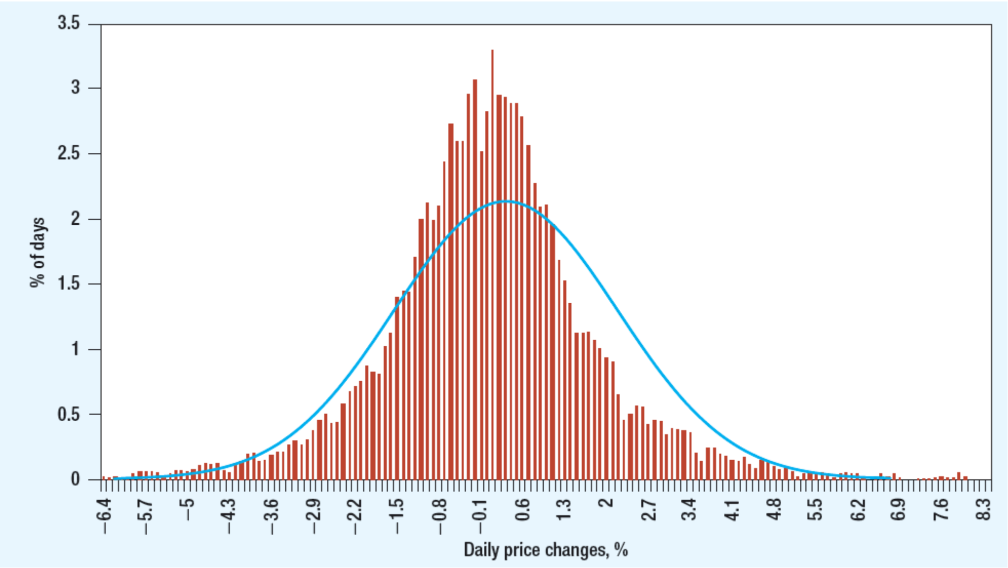
Overview

- Background
- Motivation
- Optimization problem
- Results and Conclusions
- Background Literature

Background

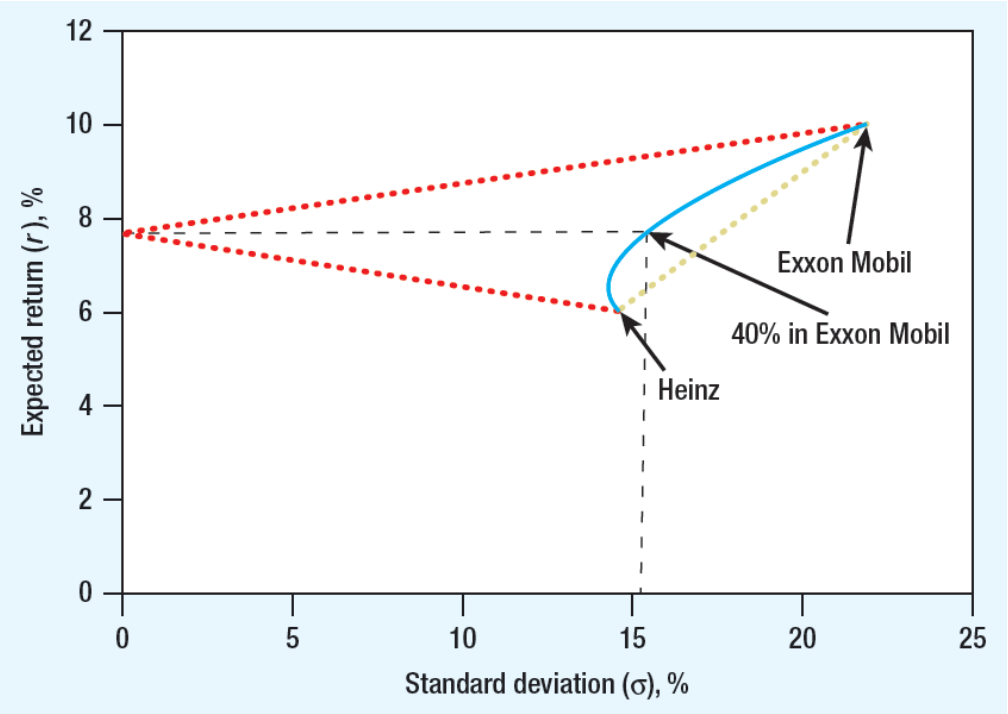
A crash course in optimal portfolio

- We keep track of the daily returns for last year.
- The histogram of daily return plots:
- **Risk** defined as standard deviation around the mean (expected **return**)
- Different stocks have different risk and return
- In this simple world, risk of multiple stocks is defined a covariance matrix.



Portfolio

- When you combine one or more stocks you get a portfolio whose risk and returns would depend upon
  - The risk and return of the components
  - The correlation between then returns of the component

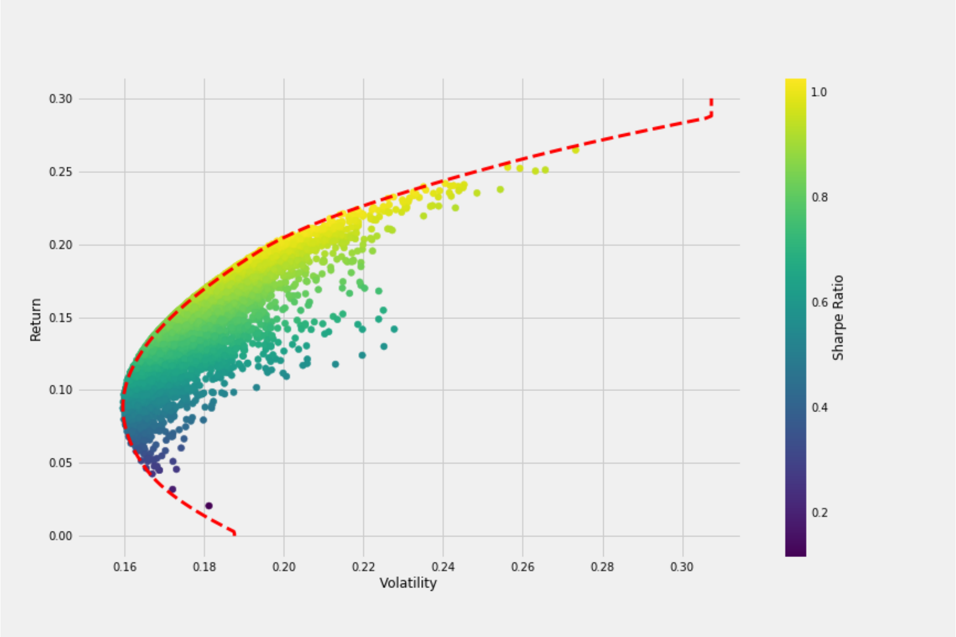


Optimal portfolio construction

- Variance of portfolio $w^T Q w$
- *w* is a vector of holding weights such that \sum w_i = 1
- Q is the covariance matrix of the returns of the assets
- Optimization gives an efficient frontier

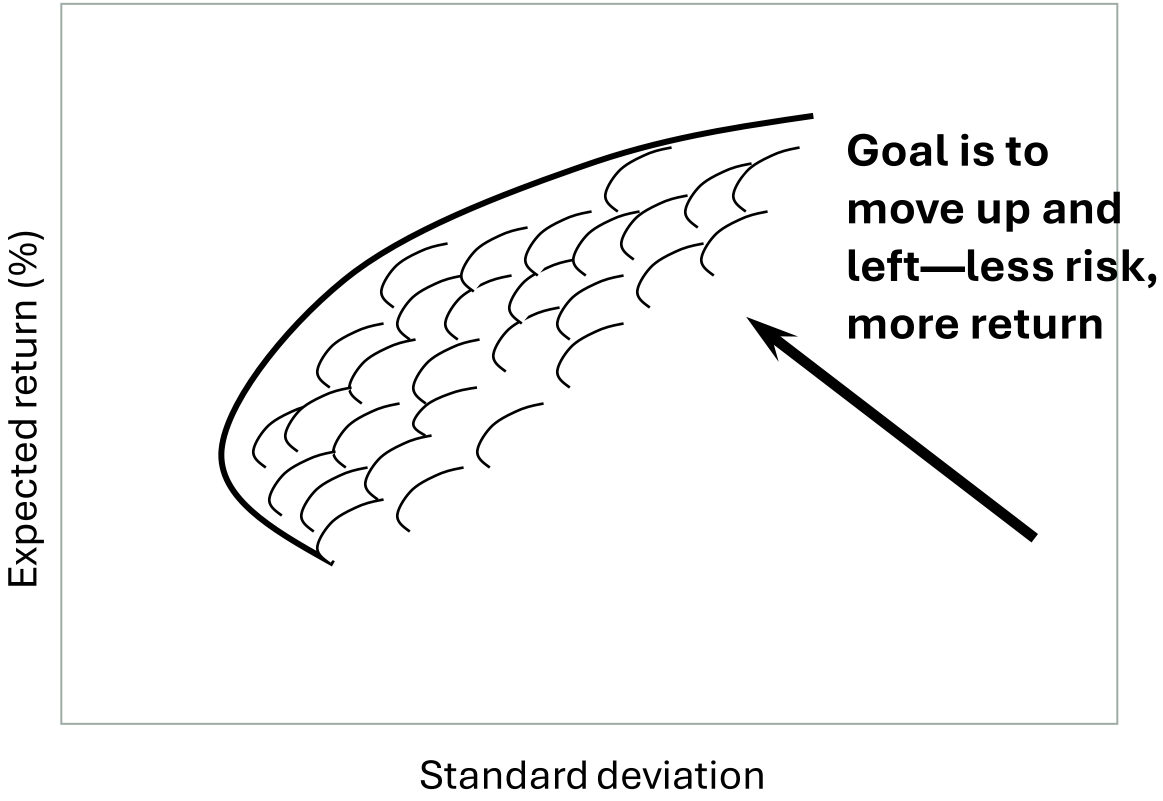
Mean-variance portfolio optimization as a quadratic program

$$\begin{aligned} & \min w^T Q w \\ \sum_{i=1}^{n} w_i \bar{r}_i & \geqslant r \\ \sum_{i=1}^{n} w_i & = 1 \\ w & \geqslant 0 \text{ if no shortsales allowed} \\ Aw & \leqslant b \text{ additional constraints} \end{aligned}$$



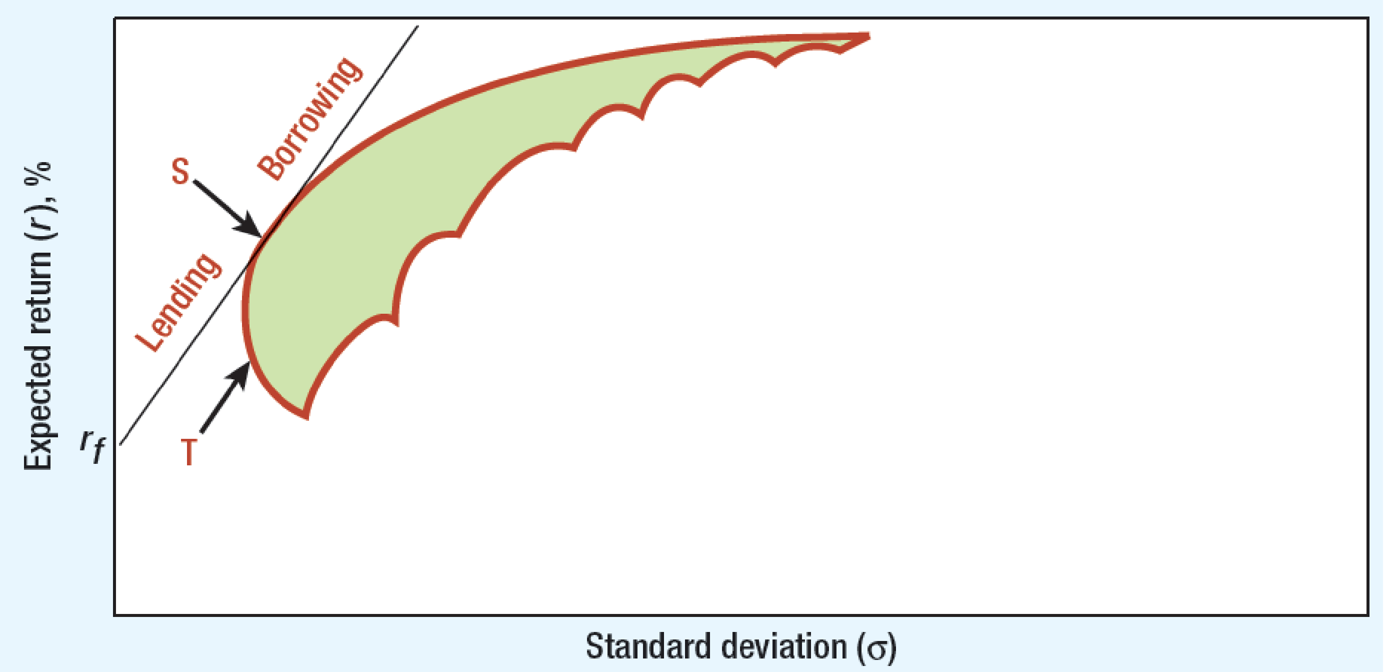
Taking optimal portfolio to extreme

- Efficient Frontier
  - Each half-ellipse represents possible weighted combinations for two stocks
  - Composite of all stock sets constitutes efficient frontier
  - The most efficient frontier would then be constructed using all possible risky assets traded in the market

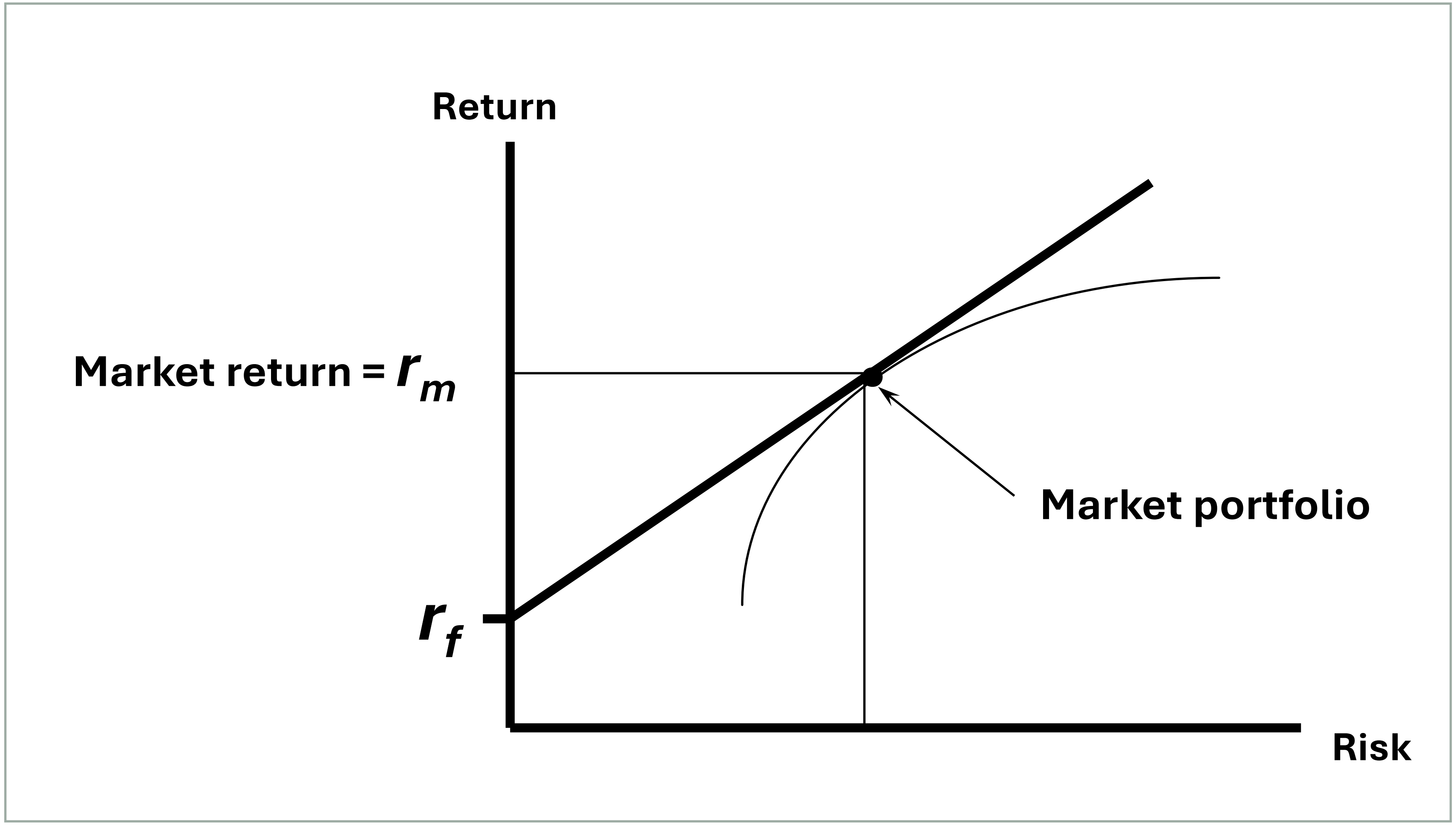


Lending and Borrowing

- We looked in a world with only risky assets
- What if we throw in a risk free asset ?
- An asset whose standard deviation of return is equal to 0?



What portfolio should I hold?



Motivation

Passive investment

- There has been empirical evidence that in the long run, investment in market index fund outperforms actively managed funds
- Outcome: A good strategy would be to hold NIFTY 50 index fund and some risk free debt instruments
- We assume risk is measured as standard deviation of returns
- In long run, maybe not appropriate
- Also the above strategy does not accommodate short term views on the index returns

Adding options

- While long term you invest in a market index fund, can you benefit in short term based on your views and return expectations on index.
- We consider constructing a portfolio which contains the underlying and market traded options written on the underlying.
- It is a single stage problem, we are time t=0, and we want to optimize returns at $T > t$.
- T is also is the maturity of the options being considered.

Options considered

- We consider call and put options written on the underlying (index) with maturity at T
- If K is the strike of the option then at T the price of the option is given by (can be evaluated only at T, when you observe S_T :
    - $V_{T,K}^{put}(S_T) = \max(K - S_T, 0)$
    - $V_{T,K}^{call}(S_T) = \max(S_T - K, 0)$
- The price of the option at t<T is given by:
    - $V_{t,K}^{put}(S_t) = \mathbb{E}^Q[V_{T,K}^{put}(S_T)|\mathcal{F}_0]$
- You can observe the price $V_{t,K}^{put}(S_t)$in the exchange for some strikes. For non-standard strikes or maturities you rely on a model.

Example pricing model

- For the GBM model, we have Black-Scholes-Merton formula:[1]

$$V_{0,K}^{\text{call}}(S) = S_0 \mathbf{\Phi}(d_+) - Ke^{-r\tau}\mathbf{\Phi}(d_-)$$

$$V_{0,K}^{\text{put}}(S) = -S_0 \mathbf{\Phi}(-d_+) + Ke^{-r\tau}\mathbf{\Phi}(-d_-)$$

Where

$\mathbf{\Phi}$ is the CDF of the standard normal distribution,

$$d_+ = \frac{\log \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

$$d_- = d_+ - \sigma\sqrt{\tau}$$

Computing the loss

- At t you construct the portfolio, and at T you unwind your portfolio
- The portfolio consists of a unit of the underlying stock bought at S_0
- w_i units of put option (maturing at T) with strike K_i bought at $w_i\ V_{0,K}^{put}(S_0), i = 1, n_1$
- w_j units of call option (maturing at T) with strike K_j bought at $w_j\ V_{0,K}^{call}(S_0), j = 1, n_2$
- Total cost of constructing the portfolio:
- $C_0 = S_0 + \sum_{i=0}^{n1} w_i V_{0,K}^{put}(S_0) + \sum_{j=0}^{n2} w_j V_{0,K}^{call}(S_0)$
- Given a realization of S_T at T the revenue at T
- $R_T = S_T + \sum_{i=0}^{n1} w_i V_{T,K}^{put}(S_T) + \sum_{j=0}^{n2} w_j V_{T,K}^{call}(S_T)$
- Loss as seen from t would be :
- $L_0 = C_0$- $R_T e^{-r_f(T-t)}$

Loss function and a risk measure

- Depending upon random variable S_T you get different loss.
- With a view on distribution of S_T we can get a distribution of Loss

Optimization

CVaR portfolio optimization

$$min_{\vec{w}} \text{ VaR} + \frac{1}{q(1-\beta)} \sum_{k=1}^{q} (\text{Loss}_k - \text{VaR})^+$$

Subject to:

$$\text{Cost} \leq C$$

$$\mathbb{E}[\text{Loss}] \leq -R$$

$$\vec{w} \geq 0$$

CVaR as LP

$$min_{\alpha,\vec{u},\vec{w}} \ \alpha + \frac{1}{q(1-\beta)} \sum_{k=1}^{q} u_k$$

Subject to:

$$\text{Cost} \leq C$$

$$\mathbb{E}[\text{Loss}] \leq -R$$

$$u_k \geq \text{Loss}_k - \alpha$$

$$\vec{u} \geq 0$$

$$\vec{w} \geq 0$$

Challenge

- Given strikes of the option, we can minimize CVaR by solving the LP.
- $CVaR(\vec{K})$ is the solution of the LP and gives the weights for each option to be held
- We also want to optimize on selecting the strikes

$$min_{\vec{K}} \mathrm{CVar}(\vec{K})$$

Subject to:

$$\mathrm{Cost} \leq C$$

$$\mathbb{E}[\mathrm{Loss}] \leq -R$$

$$u_k \geq \mathrm{Loss}_k - \alpha$$

$$\vec{u} \geq 0$$

$$\vec{w} \geq 0$$

Non-linear optimization

- We use gradient descent to optimize on strikes
- Move the strikes in the direction which minimizes the current CVaR
- We need to compute

$$\frac{\partial CVaR}{\partial K_i} \approx \frac{\delta CVaR}{\delta K_i} = \frac{CVaR(\vec{K} + \epsilon \hat{K}_i) - CVaR(\vec{K})}{\epsilon}$$

- This involves solving the LP several times (for each strike and for each update during gradient descent)

An efficient solution

A Linear program satisfies the general form

$$min_{\vec{x}} \vec{c}^T \vec{x}$$

Subject to

$$A\vec{x} \leq \vec{b}$$

$$\vec{x} \geq 0$$

and has the dual

$$max_{\vec{y}} \vec{b}^T \vec{y}$$

Subject to

$$A^T \vec{y} \leq \vec{c}$$

$$\vec{y} \geq 0$$

Theorem

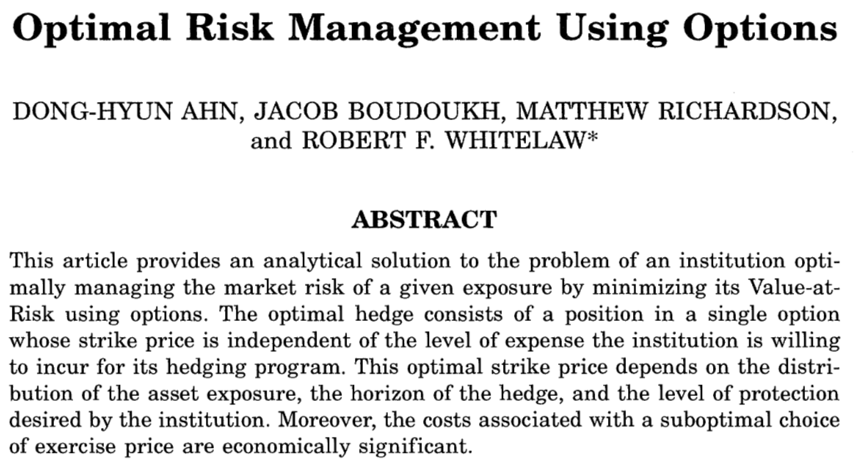
For any LP with unique primal optimal solution x^* *and unique dual optimal solution* y^**, the derivative* $\frac{\delta LP}{\delta A} = -y^* x^{*T}$

$$\frac{\delta LP}{\delta \vec{K}} = \frac{\delta LP}{\delta A} \cdot \frac{\delta A}{\delta \vec{K}}$$

Results

Results

- Classical case:
- You have a single put option, you only buy (no shorting), Black Scholes world, and you are under hedged.
- Analytical solution given by:

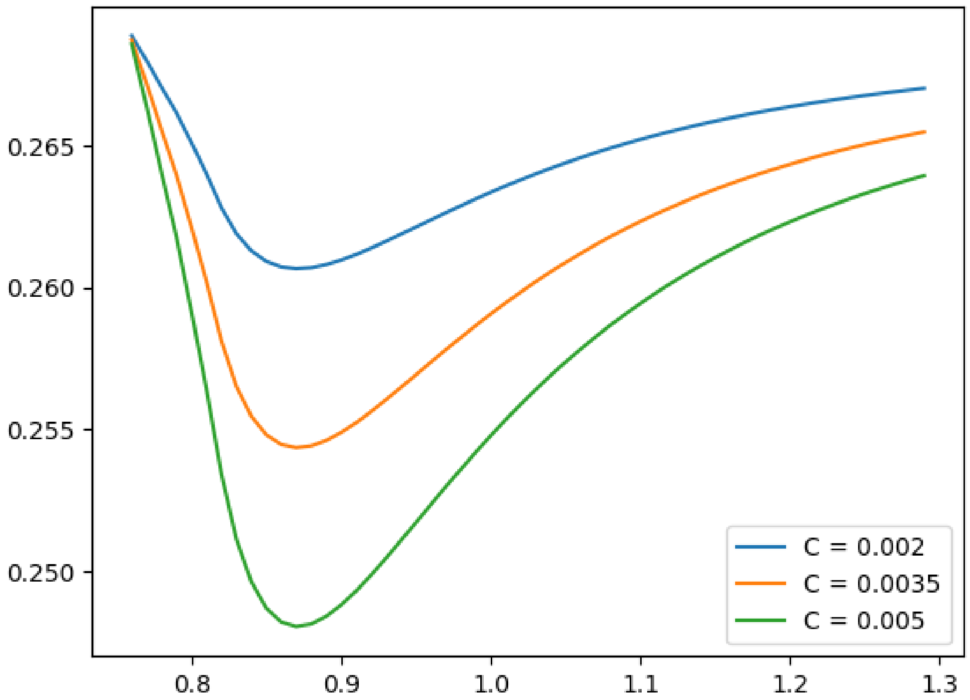


Optimal Risk Management Using Options

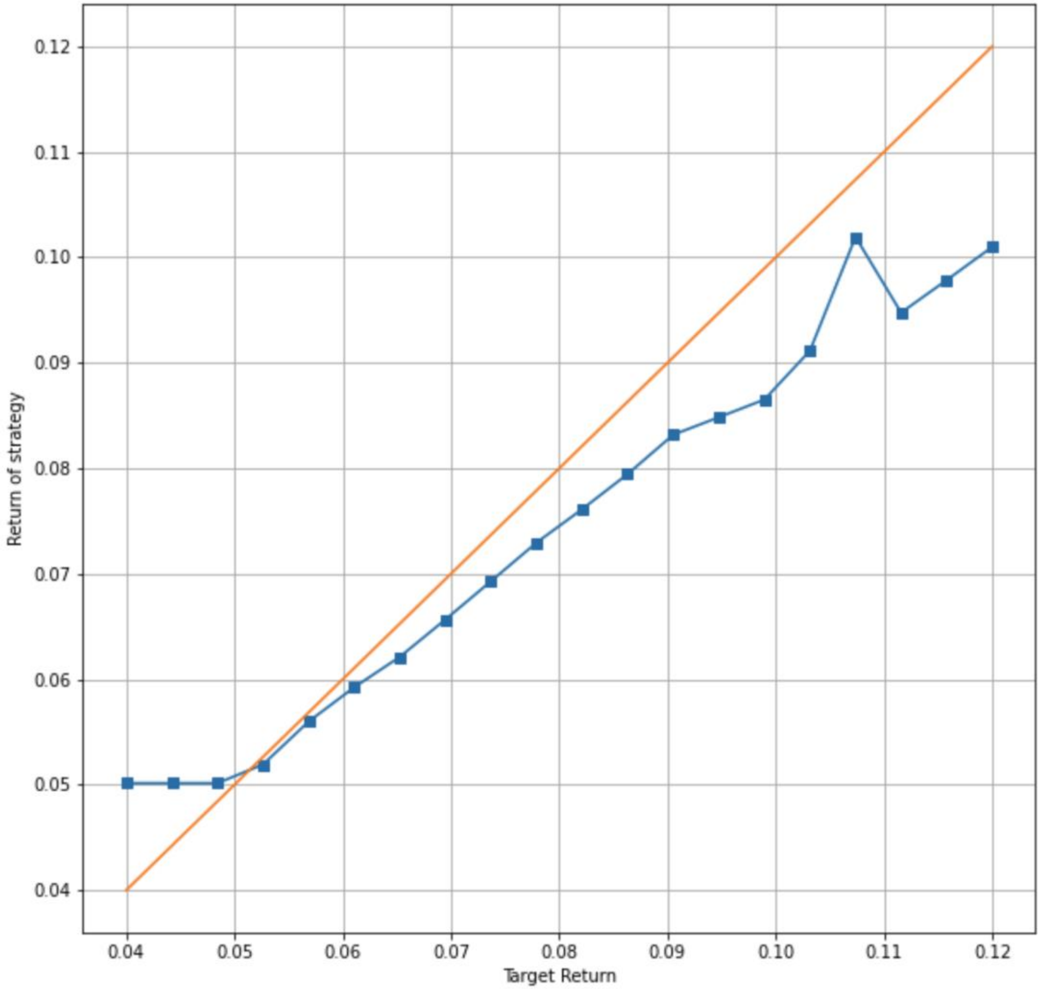
DONG-HYUN AHN, JACOB BOUDOUKH, MATTHEW RICHARDSON, and ROBERT F. WHITELAW*

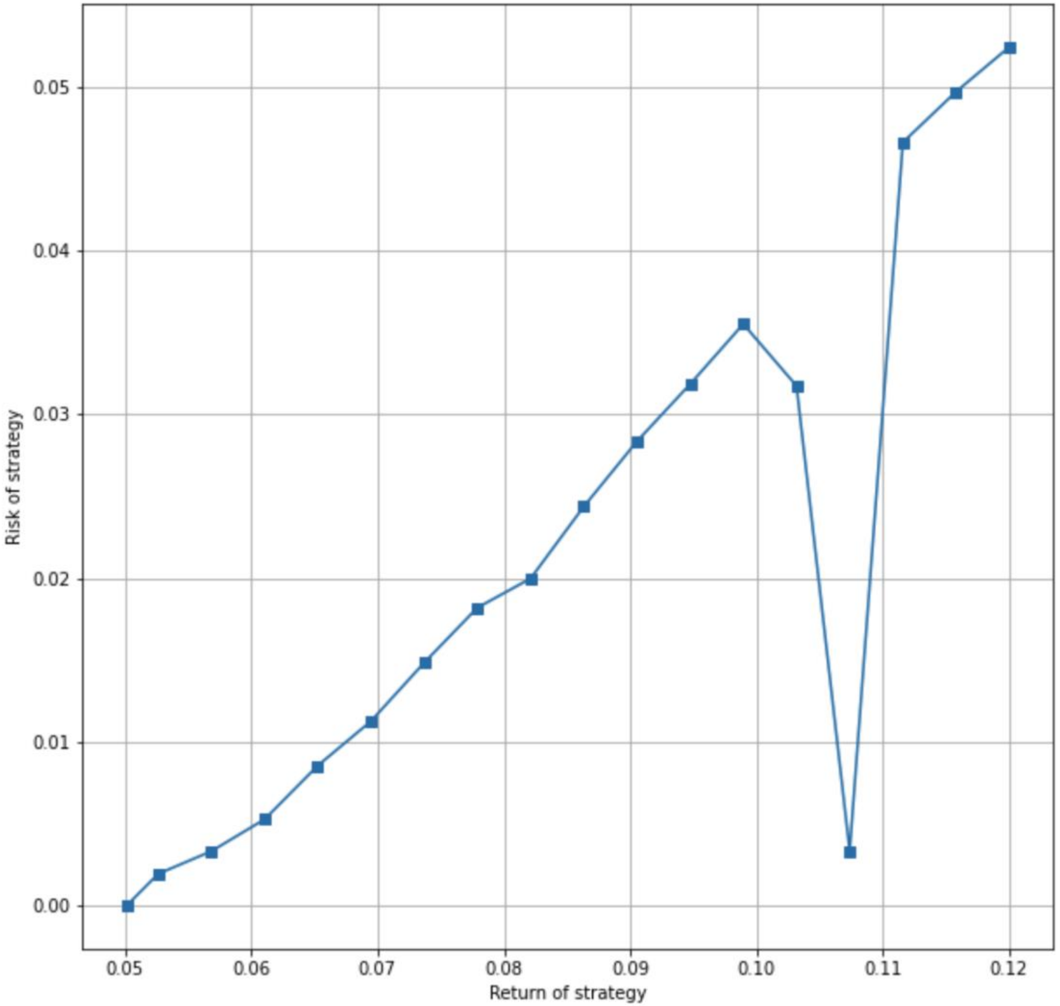
ABSTRACT

This article provides an analytical solution to the problem of an institution optimally managing the market risk of a given exposure by minimizing its Value-at-Risk using options. The optimal hedge consists of a position in a single option whose strike price is independent of the level of expense the institution is willing to incur for its hedging program. This optimal strike price depends on the distribution of the asset exposure, the horizon of the hedge, and the level of protection desired by the institution. Moreover, the costs associated with a suboptimal choice of exercise price are economically significant.



When investor demands return >= target return





Conclusion

- When you have a short-term view on index returns (distribution of returns at T)
- You can augment your passive investment in index funds with options
- The optimal portfolio will minimize extreme risk, while achieving target returns
- We have a tool with many knobs and it would be interesting to see what insights can be gained with realistic data.

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