p-th variation and roughness

Purba Das (King's college london)

Joint work with Rama Cont (Uof Oxford), Rafał Łochowski (Uof Warsaw), Toyomu Matsuda (EPFL), Nicolas Perkowsk (FU Berlin)

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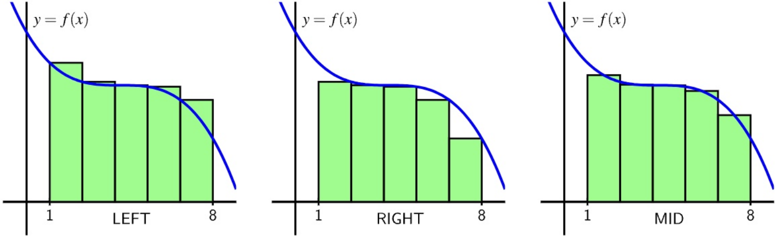

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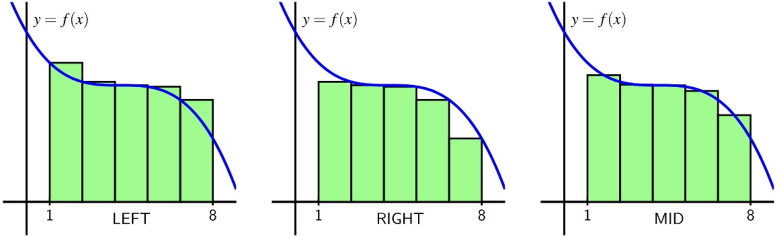

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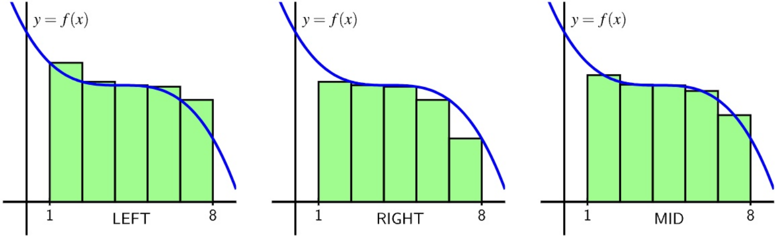

When function are ‘rough’ can we do the same? NO!!!

Sequences of partitions

A sequence of partitions π of $[0, T]$ is a sequence $(\pi^n)_{n\geq 1}$:

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Example: dyadic partition

$$\pi^n = \{0 < \frac{1}{2^n} < \frac{2}{2^n} < ... < \frac{T2^n}{2^n}\}.$$

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Example: Lebesgue partition: $t_0^n = 0$ and

$$t_{k+1}^n(\omega) = \inf\{t > t_k^n, |\omega(t) - \omega(t_k^n)| \geq T2^{-n}\}.$$

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$$[x]^{(p)}_{\pi^n}(t) = \sum_{t_j^n \in \pi^n} |x(t^n_{j+1} \wedge t) - x(t^n_j \wedge t)|^p$$

converges uniformly on $[0,T]$ to a continuous (increasing) function

$$[x]^{(p)}_{\pi} \in C^0([0,T],\mathbb{R}).$$

Quadratic variation for continuous functions

For the case of continuous function, there is an analogue definition of quadratic variation in $\mathbb{R}^d$.

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Proposition (Cont (2012))

$x \in C^0([0,T],\mathbb{R}^d)$ has finite quadratic variation along $\pi = (\pi^n, n \geq 1)$ if and only if the sequence of functions $[x]_{\pi^n}$ defined by

$$[x]_{\pi^n}(t) = \sum_{t_j^n \in \pi^n} (x(t_{j+1}^n \wedge t) - x(t_j^n \wedge t))^\top (x(t_{j+1}^n \wedge t) - x(t_j^n \wedge t))$$

converges uniformly on $[0,T]$ to a continuous (increasing) function $[x]_\pi \in C^0([0,T], S_d^+)$.

Motivation: Pathwise Itô formula

Theorem (Föllmer (1981))

Assume that $\omega \in D([0,T],\mathbb{R}^d) \cap Q_\pi([0,T],\mathbb{R}^d)$ *and* $f \in C^2(\mathbb{R}^d,\mathbb{R})$.

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Assume that $\omega \in D([0,T],\mathbb{R}^d) \cap Q_\pi([0,T],\mathbb{R}^d)$ and $f \in C^2(\mathbb{R}^d,\mathbb{R})$. Then the limit of Riemann sums:

$$\int_0^t \nabla f(\omega(s)) d^\pi \omega := \lim_{n\to\infty} \sum_{t_j^n \in \pi^n} \nabla f(\omega(t_j^n)).(\omega(t_{j+1}^n \wedge t) - \omega(t_j^n \wedge t)),$$

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exists and also one has:

$$f(\omega(t)) = f(\omega(0)) + \int_0^t \nabla f(\omega(s)) d^\pi \omega + \frac{1}{2}\int_0^t \nabla^2 f(\omega(s)) d[\omega]_\pi$$
$$+ \sum_{[0,t]} f(\omega(s)) - f(\omega(s-)) - \nabla f(\omega(s)) \Delta \omega(s).$$

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Dependence on the partition sequence

Consider now two sequences of partitions π, τ and a continuous path

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The pathwise integrals are equal if and only if $[\omega]_\pi = [\omega]_\tau$.
But the pathwise quadratic variation **does** depend on the sequence of partitions...

Quadratic variation is path dependent

The following construction (Freedman 1983) shows that the notion of pathwise quadratic variation depends on the sequence of partitions:

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- In fact Davis, Obłój and Siorpaes (2018) extend this construction to show that given *any* increasing function $A : [0,T] \to [0,\infty)$ one can construct a sequence of partitions $\pi = (\pi^n)$ such that $[\omega]_\pi(t) = A(t)$.

Motivation for intrinsic quadratic variation

- *On the other hand*, we know for Brownian paths (Dudley 1973), for *any* sequence of partitions $\pi = (\pi^n)_{n\geq 1}$ with mesh $o(1/\log n)$:

$$\mathbb{P}\left(\sum_{\pi^n} |W(t_{i+1}^n \wedge t) - W(t_i^n \wedge t)|^2 \overset{n\to\infty}{\longrightarrow} t\right) = 1.$$

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(Joint work with Rama Cont)

Balanced sequence of partitions

Let $\underline{\pi^n} = \inf_{i=0,\cdots,N(\pi^n)-1} |t_{i+1}^n - t_i^n|$ and $|\pi^n| = \sup_{i=0,\cdots,N(\pi^n)-1} |t_{i+1}^n - t_i^n|$.

Definition

We say a sequence of partitions $\pi = (\pi^n)_{n\geq 1}$ balanced if

$$\exists c > 0, \qquad \forall n \geq 1, \quad \frac{|\pi^n|}{\underline{\pi^n}} \leq c. \tag{1}$$

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This condition means that the intervals in the partition π^n are asymptotically comparable.

Definition (Quadratic roughness)

Let $\mathbb{T} = (\mathbb{T}^n)_{n \geq 1}$ be the dyadic (Reference) partition of $[0, T]$ and $\pi^n = \left(0 = s_0^n < s_1^n < \cdots < s_{N(\pi^n)}^n = T\right)$ be a balanced sequence of partitions of $[0, T]$ with vanishing mesh $|\pi^n| \to 0$. We say that $x \in C^0([0, T], \mathbb{R}^d) \cap Q_{\mathbb{T}}([0, T], \mathbb{R}^d)$ has the quadratic roughness property with coarsening index $0 < \beta < 1$ along π on $[0, T]$ if there exists a subsequence or super-sequence $d^n = \left(0 = t_1^n < t_2^n < \cdots < t_{N(d^n)}^n = T\right)$ of $\mathbb{T}$ with the following properties:

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(i) $|d^n|^\beta = O\left(|\pi^n|\right)$, and

(ii) for all $t \in [0,T]$:

$$\sum_{j=1}^{N(\pi^n)-1} \sum_{t_i^n \neq t_{i'}^n \in (s_j^n, s_{j+1}^n]} \left(x(t_{i+1}^n \wedge t) - x(t_i^n \wedge t)\right)^t \left(x(t_{i'+1}^n \wedge t) - x(t_{i'}^n \wedge t)\right) \xrightarrow{n\to\infty} 0.$$

We denote by $R_\pi^\beta([0,T],\mathbb{R}^d)$ the set of paths satisfying this property.

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- Note that, since $\beta < 1$, the number of terms in the inner sum in (ii) grows to infinity as n grows, so (ii) is the result of compensation across terms, reminiscent of the law of large numbers.
- This quadratic roughness plays a crucial role in the stability of quadratic variation with respect to the partitions.

Choice of reference partition

The dyadic partition may be replaced by any other balanced sequence of partitions σ with vanishing mesh $|\sigma^n| \to 0$ satisfying

$$\sup_n \frac{|\sigma^n|}{|\sigma^{n+1}|} < \infty$$

without changing any of the statements of the results.

Quadratic roughness for Brownian paths

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Theorem (Quadratic roughness of Brownian paths)

Let W be a Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $T > 0$ and $(\pi^n)_{n\geq 1}$ a balanced sequence of partitions of $[0, T]$ with

$$(\log n)^2 |\pi^n| \overset{n\to\infty}{\to} 0. \tag{2}$$

Then, for any $0 < \beta < 1$, the sample paths of W almost-surely satisfy the quadratic roughness property with coarsening index β:

$$\forall \beta \in (0,1), \quad \mathbb{P}\left(W \in R_\pi^\beta([0,T], \mathbb{R}) \right) = 1.$$

Necessicity of quadratic roughness

The quadratic roughness property is a necessary condition for the stability of quadratic variation with respect to the choice of partition sequence.

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Lemma (Cont & Das 2022)

Let $x \in C^{\alpha}([0,T],\mathbb{R}^d) \cap Q_{\mathbb{T}}([0,T],\mathbb{R}^d)$. *Let* $\pi = (\pi^n)_{n\geq 1}$ *be a balanced partition sequence of* $[0,T]$ *such that* $x \in Q_{\pi}([0,T],\mathbb{R}^d)$. *Then:*

$$(\forall t \in [0,T]),\ [x]_{\pi}(t) = [x]_{\mathbb{T}}(t)\ ) \quad \Rightarrow \quad \forall \beta \in (0,2\alpha),\ x \in R^{\beta}_{\pi}([0,T],\mathbb{R}^d).$$

Uniqueness of pathwise QV for rough functions

Our main result is that quadratic roughness along such a sequence implies the uniqueness of pathwise quadratic variation:

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Theorem (Uniqueness of quadratic variation (Cont & Das 2022))

Let π *be a balanced sequence of partitions of* $[0,T]$ *and* $x \in C^{\alpha}([0,T],\mathbb{R}^d) \cap R^{\beta}_{\pi}([0,T],\mathbb{R}^d)$ *for some* $0<\beta<2\alpha$*. Then*

$$x \in Q_\pi([0,T],\mathbb{R}^d), \qquad \text{and} \qquad \forall t \in [0,T], \quad [x]_\pi(t) = [x]_{\mathbb{T}}(t).$$

Invariant definition of quadratic variation

Let $\mathbb{T} = (\mathbb{T}^n)_{n\geq 1}$ be the dyadic sequence of partitions of $[0,T]$. Define,

$$\mathcal{Q}([0,T],\mathbb{R}^d) = C^{\frac{1}{2}-}([0,T],\mathbb{R}^d) \cap Q_{\mathbb{T}}([0,T],\mathbb{R}^d). \tag{3}$$

Lemma

The class $\mathcal{Q}([0,T],\mathbb{R}^d)$ is non-empty and contains all 'typical' Brownian paths.

Quadratic variation map [Cont & Das 2022]

We have the following invariant quadratic variation map.

Theorem (Quadratic variation map)

There exists a unique map:

$$\begin{aligned} [\,.\,] \quad : \mathcal{Q}([0,T],\mathbb{R}^d) &\to C^0([0,T],S_d^+) \\ x &\to [x] \end{aligned}$$

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such that: $\forall \pi \in \mathbb{B}([0,T]), \quad \forall \beta \in (0,1),$
$\forall x \in R_\pi^\beta([0,T],\mathbb{R}^d) \cap \mathcal{Q}([0,T],\mathbb{R}^d), \quad \forall t \in [0,T],$ *we get:*

$$[x]_\pi(t) = [x](t).$$

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$$[x]_\pi(t) = [x](t).$$

We call $[x]$ *the quadratic variation of* x*.*

Theorem (Invariance of the Föllmer integral (Cont & Das 2022))

There exists a unique map

$$\begin{aligned} I \quad : C^2(\mathbb{R}^d) \times Q([0,T],\mathbb{R}^d) &\rightarrow Q([0,T],\mathbb{R}) \\ (f,x) &\rightarrow I(f,x) = \int_0^{\cdot} (\nabla f \circ x).dx, \end{aligned}$$

Pathwise integration and pathwise Itô formula

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$$I(f,x)(t) = \int_0^t (\nabla f \circ x).d^\pi x = \lim_{n\to\infty} \sum_{\pi^n} \nabla f(x(t_i^n)).(x(t_{i+1}^n \wedge t) - x(t_i^n \wedge t)).$$

We denote $I(f,x) = \int_0^{\cdot} (\nabla f \circ x) dx.$

Pathwise integration and Itô formula (cont.)

Theorem (Pathwise change of variable formula)

$\forall f \in C^2(\mathbb{R}^d), \quad \forall \pi \in \mathbb{B}([0,T]), \quad \forall \beta \in (0,1)$, *and*, $\forall x \in R_\pi^\beta([0,T],\mathbb{R}^d) \cap \mathcal{Q}([0,T],\mathbb{R}^d)$, *we have the following change of variable formula:*

$$f(x(t)) - f(x(0)) = \int_0^t (\nabla f \circ x).dx + \frac{1}{2}\int_0^t < \nabla^2 f(x), d[x] >$$

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$$\forall \omega \in \Omega_\pi, \ [\omega]_\pi(t) = t.$$
6. On the other hand, we know from Freedman's result there exists for each $\omega \in \Omega$ a partition $\pi = \pi(\omega)$ such that $[\omega]_{\pi(\omega)}(t) = 0$, and therefore
$$\cap_\pi \Omega_\pi = \emptyset.$$

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3. Balanced partition π + quadratic roughness on path $x \implies [x]_\pi = [x]_{\text{ref. part.}}$.
4. Brownian motion satisfies this quadratic roughness property almost surely.
5. In fact, for any *deterministic* partition sequence $\pi = (\pi^n)$ with $|\pi^n| \log n \to 0$, there exists $\Omega_\pi \subset \Omega$ of full $\mathbb{P}$-measure such that
$$\forall \omega \in \Omega_\pi, \ [\omega]_\pi(t) = t.$$
6. On the other hand, we know from Freedman's result there exists for each $\omega \in \Omega$ a partition $\pi = \pi(\omega)$ such that $[\omega]_{\pi(\omega)}(t) = 0$, and therefore
$$\cap_\pi \Omega_\pi = \emptyset.$$

So even for Brownian motion, quadratic roughness does not ensure an almost sure invariance of quadratic variation across all deterministic partitions (partitions purely on time variable).

Some obvious questions!!

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(Joint work with Rafał Łochowski, Toyomu Matsuda & Nicolas Perkowski)

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- Unlike in Dudley's result, there is no condition on the decay of meshes of partitions and the null set is uniform over all (uniform) Lebesgue partitions.

fractional Brownian motion analogue

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Theorem (Fractional analogue of Chacon et al. (D-L-M-P 2023))

Let $H < 1/2$ and let $\mathfrak{c}_H$ be a constant. Then, there exists a measurable set $\Omega_H \subseteq C([0, \infty); \mathbb{R})$ with the following property.

- $\mathbb{P}(B^H \in \Omega_H) = 1.$
- *For every $w \in \Omega_H$ and $T \in (0, \infty)$, we have*

$$\lim_{\epsilon \to 0, \epsilon > 0} \sup_{\substack{\mathbb{L}:|\mathbb{L}| \leq \epsilon, \\ t \leq T}} |V_{0,t}(\mathbb{L}, w) - \mathfrak{c}_H t| = 0.$$

A non-intuitive Conjecture

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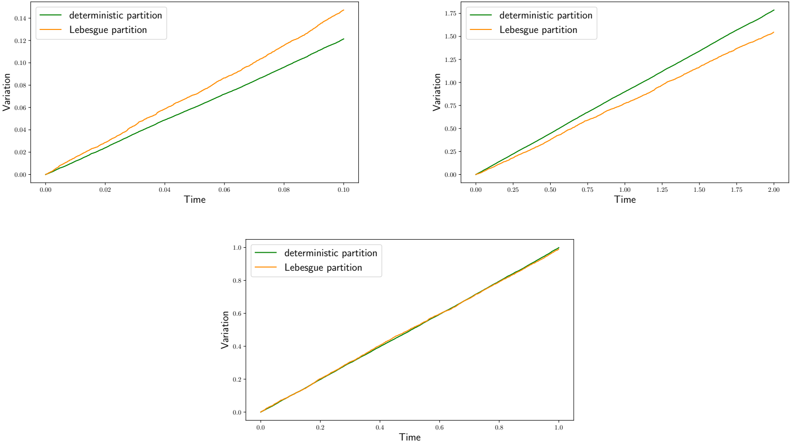

Figure: $1/H$-th variation of fBM. $H = 0.4, 0.6, 0.5$ respectively.

Horizontally rough

We denote by $K_{s,t}(\epsilon, w)$ the number of ϵ-level crossings in the interval $[s, t]$

Definition (Horizontally rough: an invariance notion for p-th var)

A function $x \in C^0([0,T],\mathbb{R})$ is called *horizontally rough* if for any $t \in [0,T]$, $\rho \in \mathbb{R}$ and $\epsilon = \{\epsilon_n\}$ with $\epsilon_n \downarrow 0$,

$$\lim_{n\to\infty} \frac{K_{0,t}(\epsilon_n, x+\rho)}{K_{0,t}(\epsilon_n, x)} = 1.$$

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- Using results from Chacon et al.(1981) one can show that Brownian motion and more generally continuous semimartingales are horizontally rough almost surely.
- Our result shows that fractional Brownian motion with Hurst index $H < 1/2$ is horizontally rough almost surely.