Higher Order Time Discretization For The Stochastic Semilinear Wave Equation

With Andreas Prohl (Tübingen) and Xiaobing Feng (Tennessee) *

Akash Ashirbad Panda
(Assistant Professor, IIT Bhubaneswar)

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Contents

The Stochastic Model

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- Let $\mathfrak{P} := (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$, and $\{W(t)\}_{t \geq 0}$ be a finite dimensional Wiener process defined on it.
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We investigate the **numerical approximation** of the following stochastic wave equation perturbed by multiplicative noise of Itô type:

The Stochastic Semilinear Wave Equation

$$\begin{cases} \partial_t^2 u - \Delta u = F(u, \partial_t u) + \boxed{\sigma(u, \partial_t u)\,\partial_t W} & \text{in } (0,T)\times D\,, \\ u(0,\cdot) = u_0\,, \quad \partial_t u(0,\cdot) = v_0 & \text{in } D\,, \\ u(t,\cdot) = 0 & \text{on } \partial D,\ \forall\, t\in(0,T)\,, \end{cases} \tag{1}$$

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- u denotes the displacement/position, $\partial_t u$ denotes the velocity;
- Here, $F(\cdot, \cdot)$ and $\sigma(\cdot, \cdot)$ are Lipschitz in both arguments;
- The initial data u_0 and v_0 are given $\mathcal{F}_0$-measurable random variables.

The Problem of Interest: The Strong Approximation

Considering a numerical scheme of a stochastic equation

Types of Errors

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- The strong error measures the pathwise approximation of the true solution by a numerical one.
- The weak order of convergence is concerned with the approximation of the law of the solution at a fixed time.
- We discuss the strong approximation of (1), *i.e.,*

$$\boxed{\mathbb{E}\Big[\|u(n\Delta t,\cdot) - u^n\|_{\mathbb{L}^2}^2\Big] \leq C\big(\Delta t\big)^{\delta}}$$

- If such a bound is true, we say that the numerical scheme has strong order of convergence δ or strong rates of convergence δ.

First Order System and Total Energy

Chow [2] (2015)

A strong variational solution to (1) exists, and is usually constructed via the reformulation of $(1)_1$ as a first order system by setting $\boxed{v = \partial_t u}$,

$$\begin{cases} \mathrm{d}u = v\,\mathrm{d}t \\ \mathrm{d}v = \left[\Delta u + F(u,v)\right]\mathrm{d}t + \sigma(u,v)\,\mathrm{d}W(t)\,. \end{cases} \tag{2}$$

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We associate the following energy functional

$$\mathcal{E}(u, v) := \underbrace{\frac{1}{2}\int_{\mathcal{O}} |\nabla u(x)|^2 \mathrm{d}x}_{\text{Elastic Energy}} + \underbrace{\frac{1}{2}\int_{\mathcal{O}} |v(x)|^2 \mathrm{d}x}_{\text{Kinetic Energy}}\,. \tag{3}$$

Why Study with $F(u, v)$ and $\sigma(u, v)$?

Why such a system is of importance? (Dalang [4] (2009))

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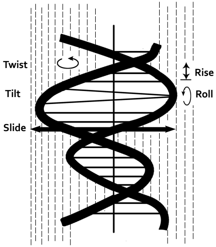
Many biological events are related to the motion of the DNA string; for instance, an enzyme may be released.



- A DNA molecule can be viewed as a long elastic string, whose length is essentially infinitely long compared to its diameter.
- A DNA molecule floats in a fluid, so it is constantly in motion, just as a particle of pollen floating in a fluid moves according to Brownian motion.

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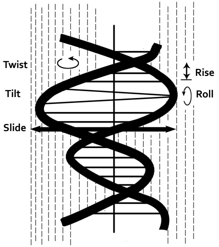

- A DNA molecule can be viewed as a long elastic string, whose length is essentially infinitely long compared to its diameter.
- A DNA molecule floats in a fluid, so it is constantly in motion, just as a particle of pollen floating in a fluid moves according to Brownian motion.
- The forces acting on the string are mainly of three kinds:
  1. elastic forces, which include torsion forces,
  2. **friction** due to viscosity of the fluid;
  3. random impulses due the the impacts on the string of the **fluid's molecules**.

The Effect of Noise

Example 1

Let $D = (0,1)$, $T = 1$, $F \equiv 0$ in (1), and W be of the form

$$W(t,x,\omega) := \sum_{j=1}^{M} \beta_j(t,\omega) e_j(x)\,, \tag{4}$$

where $\{\beta_j(t,\omega); t \geq 0\}$ are mutually independent Brownian motions and $e_j(x) = \sqrt{2}\sin(j\pi x)$. Let $u_0(x) = \sin(2\pi x)$ and $v_0(x) = \sin(3\pi x)$.

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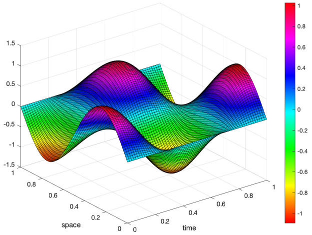


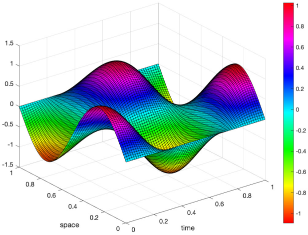
$\text{Case 1}: \sigma(u,v) = 0$

Example 2

Let $\mathcal{O} = (0,1)$, $T = 1$, $F \equiv 0$ in (1), and W be of the form

$$W(t,x,\omega) := \sum_{j=1}^{M} \beta_j(t,\omega) e_j(x)\,, \tag{5}$$

where $\{\beta_j(t,\omega); t \geq 0\}$ are mutually independent Brownian motions and $e_j(x) = \sqrt{2}\sin(j\pi x)$. Let $u_0(x) = \sin(2\pi x)$ and $v_0(x) = \sin(3\pi x)$.



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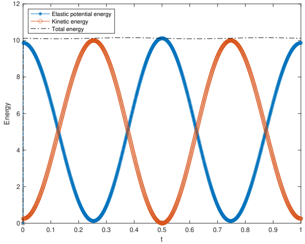


Case 1, Energy Curves

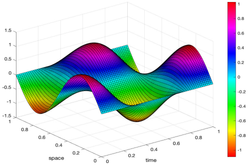


Fig.-1 : Case 1 : $\sigma(u, v) = 0$

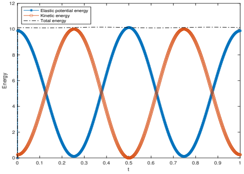


Fig.-2 : Case 1, Energy Curves

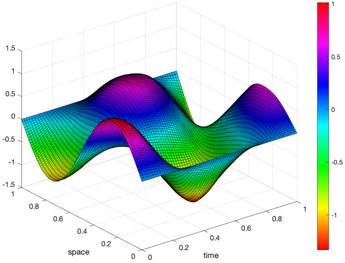


Fig.-3 : Case 2 : $\sigma(u, v) = u$

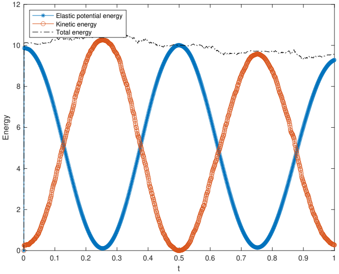
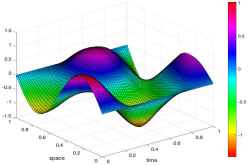


Fig.-4 : Case 2, Energy Curves

Effect of Noise



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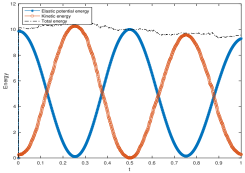

Fig.-4 : Case 2, Energy Curves

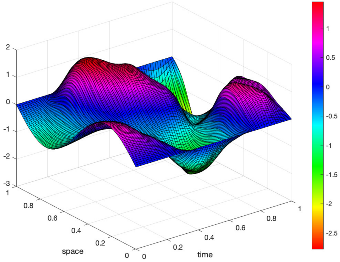

Fig.-5 : Case 3 : $\sigma(u, v) = v$

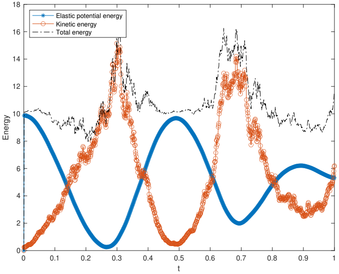

Fig.-6 : Case 3, Energy Curves

What Happens to the Approximate Total Energy, i.e., Plots $t \mapsto \mathbb{E}_{\mathrm{MC}}[\mathcal{E}(u(t), v(t))]$

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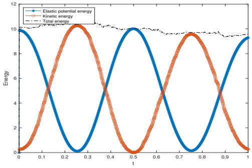

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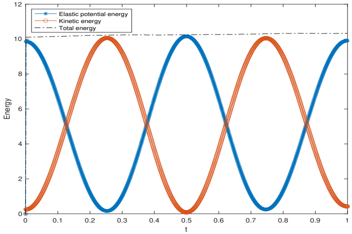
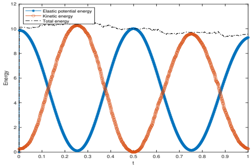


$t \mapsto \mathbb{E}_{\mathrm{MC}}[\mathcal{E}(u(t), v(t))]$, with $\mathrm{MC} = 10^3$



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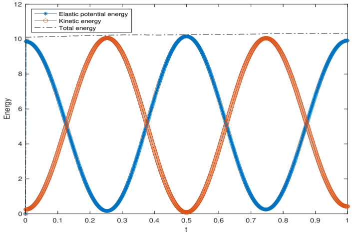
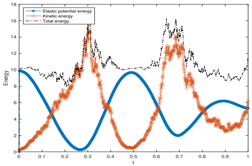


$t \mapsto \mathbb{E}_{MC}[\mathcal{E}(u(t), v(t))]$, with $MC = 10^3$



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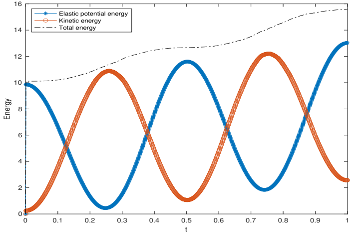

$t \mapsto \mathbb{E}_{MC}[\mathcal{E}(u(t), v(t))]$, with $MC = 5 \times 10^3$

The Highlight of The Work: 3 Important Results

Aim of our work

We focus on the proper time discretization (which we consider to be the essential part of an overall discretization) for the SPDE:

$$\begin{cases} \mathrm{d}u = v\,\mathrm{d}t \\ \mathrm{d}v = \left[\Delta u + F(u,v)\right]\mathrm{d}t + \sigma(u,v)\,\mathrm{d}W(t)\,. \end{cases} \tag{6}$$

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We address the following problems:

Result 1:- For the Case: $F \equiv F(u, v)$ and $\sigma \equiv \sigma(u, v)$

We use an implicit method in time to approximate (6), and we obtained

$$\boxed{\mathcal{O}(k^{\frac{1}{2}}) \text{ for the temporal error}}$$

in this general case, where $\{t_n\}_{n=0}^{N}$ be a mesh of size $\Delta t = k > 0$ covering $[0, T]$. This has not been addressed in the existing literature.

The Highlight of our Work: for $\sigma \equiv \sigma(u)$, $F \equiv F(u)$

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Result 2:- For the case $\sigma \equiv \sigma(u)$, $F \equiv F(u)$

We use energy arguments to obtain

$$\boxed{\mathcal{O}(k) \text{ for the temporal error}}$$

This coincides with the order obtained in Anton *et al.* [1] (2016) and Cohen *et al.* [3] (2016).

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Result 3:- For the case $\sigma \equiv \sigma(u)$, $F \equiv F(u)$

With the introduction of an additional term to our scheme, we improve it to a higher-order scheme which yields

improved convergence order $\mathcal{O}(k^{3/2})$ for approximates of u

The Numerical Scheme

Numerical Scheme

$(\widehat{\alpha}, \beta)-$scheme

Fix $\widehat{\alpha} \in \{0, 1\}$ and $\beta \in [0, 1/2)$. Let $\{(u^n, v^n)_{n=0,1}\}$ be given $\mathcal{F}_{t_n}$-measurable, $[\mathbb{H}_0^1]^2$-valued r.v's. For every $n \geq 1$, find $[\mathbb{H}_0^1]^2$-valued, $\mathcal{F}_{t_{n+1}}$-measurable r.v's (u^{n+1}, v^{n+1}) such that $\mathbb{P}$-a.s.

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$$(u^{n+1} - u^n, \phi) = \Delta t (v^{n+1}, \phi) \quad \forall \phi \in \mathbb{L}^2 , \tag{8}$$

$$\begin{aligned}(v^{n+1} - v^n, \psi) = -\Delta t (\nabla \widetilde{u}_\beta^{n,\frac{1}{2}}, \nabla \psi) + \Big(\sigma(u^n, v^{n-\frac{1}{2}})\, \Delta_n W, \psi\Big) \\ + \boxed{\widehat{\alpha} \Big(D_u \sigma(u^n, v^{n-\frac{1}{2}})\, v^n\, \widetilde{\Delta_n W}, \psi\Big)} \\ + \frac{\Delta t}{2} \Big(3F(u^n, v^n) - F(u^{n-1}, v^{n-1}), \psi\Big) \quad \forall \psi \in \mathbb{H}_0^1 ,\end{aligned} \tag{9}$$

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where

$$\widetilde{u}_\beta^{n,\frac{1}{2}} := \frac{1 + \beta\,(\Delta t)^\beta}{2} u^{n+1} + \frac{1 - \beta\,(\Delta t)^\beta}{2} u^{n-1}\,, \tag{10}$$

and

$$\Delta_n W := W(t_{n+1}) - W(t_n) \quad \text{and} \quad v^{n-\frac{1}{2}} := \frac{1}{2}(v^n + v^{n-1})\,.$$

$$(u^{n+1} - u^n, \phi) = \Delta t (v^{n+1}, \phi) \quad \forall \phi \in \mathbb{L}^2 , \tag{11}$$

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where

$$\widetilde{\Delta_n W} := \int_{t_n}^{t_{n+1}} (s - t_n)\, \mathrm{d}W(s) = \int_{t_n}^{t_{n+1}} s\, \mathrm{d}W(s) - t_n \Delta_n W . \tag{13}$$

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By Itô's formula, we can rewrite $\widetilde{\Delta_n W}$ as

$$\boxed{\widetilde{\Delta_n W} = \int_{t_n}^{t_{n+1}} \big[W(t_{n+1}) - W(s)\big]\, \mathrm{d}s = kW(t_{n+1}) - \int_{t_n}^{t_{n+1}} W(s)\, \mathrm{d}s.} \tag{14}$$

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$$\begin{aligned}(v^{n+1} - v^n, \psi) = -\Delta t\big(\nabla \widetilde{u}_\beta^{n,\frac{1}{2}}, \nabla\psi\big) &+ \Big(\sigma(u^n, v^{n-\frac{1}{2}})\,\Delta_n W, \psi\Big) \\ &+ \widehat{\alpha}\Big(D_u\sigma(u^n, v^{n-\frac{1}{2}})\, v^n\, \widetilde{\Delta_n W}, \psi\Big) \\ &+ \frac{\Delta t}{2}\Big(3F(u^n, v^n) - F(u^{n-1}, v^{n-1}), \psi\Big) \quad \forall \psi \in \mathbb{H}_0^1 \,,\end{aligned} \tag{12}$$

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$$\widetilde{\Delta_n W} := \int_{t_n}^{t_{n+1}} (s - t_n)\,\mathrm{d}W(s) = \int_{t_n}^{t_{n+1}} s\,\mathrm{d}W(s) - t_n \Delta_n W \,. \tag{13}$$

By Itô's formula, we can rewrite $\widetilde{\Delta_n W}$ as

$$\boxed{\widetilde{\Delta_n W} = \int_{t_n}^{t_{n+1}} \big[W(t_{n+1}) - W(s)\big]\,\mathrm{d}s = kW(t_{n+1}) - \int_{t_n}^{t_{n+1}} W(s)\,\mathrm{d}s.} \tag{14}$$

The β-term

The 'β-term' is necessary for the stability analysis, in order to handle the noise term, which is unlike any *parabolic* SPDEs.

Higher Order Scheme when $F \equiv F(u)$, $\sigma \equiv \sigma(u)$

Let's recall the notation

$$\widetilde{u}_{\beta}^{n,\frac{1}{2}} := \frac{1+\beta\, k^{\beta}}{2} u^{n+1} + \frac{1-\beta\, k^{\beta}}{2} u^{n-1}\,, \tag{15}$$

For $\beta = 0$,

$$\boxed{\widetilde{u}_{\beta}^{n,\frac{1}{2}} = u^{n,\frac{1}{2}} = \frac{1}{2}(u^{n+1} + u^{n-1})}.$$

This is inspired by the second order time-stepping scheme of Dupont [6] (1973) for the deterministic wave equation.

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Also, in the case when $\boxed{F \equiv F(u),\ \sigma \equiv \sigma(u) \text{ and } \beta = 0,}$, the $(\widehat{\alpha}, \beta)-$scheme simplifies to (for $n \geq 1$)

$(\widehat{\alpha}, 0)-$scheme

$$(u^{n+1} - u^n, \phi) = \Delta t(v^{n+1}, \phi) \quad \forall \phi \in \mathbb{L}^2, \tag{16}$$

$$\begin{aligned}(v^{n+1} - v^n, \psi) &= -\Delta t\big(\nabla u^{n,\frac{1}{2}}, \nabla\psi\big) + \Big(\sigma(u^n)\Delta_n W, \psi\Big) \\ &\quad + \widehat{\alpha}\Big(D_u\sigma(u^n)v^n\widetilde{\Delta_n W}, \psi\Big) \\ &\quad + \frac{\Delta t}{2}\Big(3F(u^n) - F(u^{n-1}), \psi\Big) \quad \forall \psi \in \mathbb{H}_0^1.\end{aligned} \tag{17}$$

Numerical Example - Comparison between the cases $\widehat{\alpha} = 0$ and $\widehat{\alpha} = 1$

Example 3

Let $D = (0,1)$, $T = 1$, $F \equiv 0$, $\sigma(u) = \sin(u)$ in the equation (7). Let

$$u_0(x) = \sin(2\pi x) \qquad \text{and} \qquad v_0(x) = \sin(3\pi x)\,,$$

and W as in Example 2.

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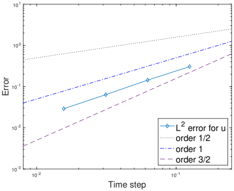
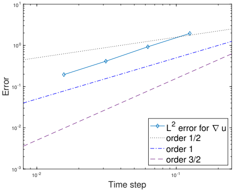
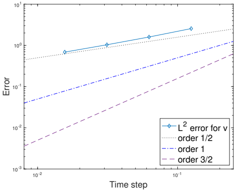


(a) $\mathbb{L}^2$-error for u



(b) $\mathbb{L}^2$-error for ∇u



(c) $\mathbb{L}^2$-error for v

Figure 1: **(Example 3)** Temporal rates of convergence; discretization parameters: $h = 2^{-7}, \Delta t = \{2^{-3}, \cdots, 2^{-6}\}$, MC = 3000.

(1, 0)-scheme, i.e., $\widehat{\alpha} = 1$, $\beta = 0$

$$(u^{n+1} - u^n, \phi) = \Delta t(v^{n+1}, \phi) \quad \forall \phi \in \mathbb{L}^2\,, \tag{18}$$

$$\begin{aligned}(v^{n+1} - v^n, \psi) = & -\Delta t\Big(\nabla u^{n,\frac{1}{2}}, \nabla\psi\Big) + \Big(\sigma(u^n)\Delta_n W, \psi\Big) \\ & + \widehat{\alpha}\left(D_u\sigma(u^n)v^n\widetilde{\Delta_n W}, \psi\right) \\ & + \frac{\Delta t}{2}\Big(3F(u^n) - F(u^{n-1}), \psi\Big) \quad \forall \psi \in \mathbb{H}^1_0\,.\end{aligned} \tag{19}$$

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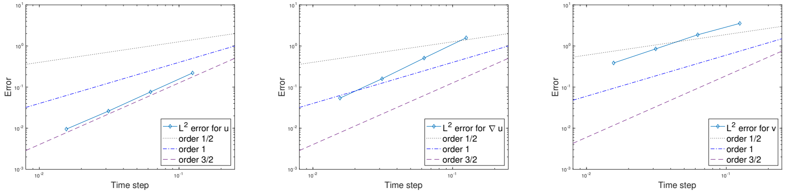

(a) $\mathbb{L}^2$-error for u (b) $\mathbb{L}^2$-error for ∇u (c) $\mathbb{L}^2$-error for v

Figure 2: **(Example 3)** Temporal rates of convergence; discretization parameters: $h = 2^{-7}, \Delta t = \{2^{-3}, \cdots, 2^{-6}\}$, MC $= 3000$.

A Computable Approximation of $\widetilde{\Delta_n W}$

Recall that by Itô's formula,

$$\widetilde{\Delta_n W} = \int_{t_n}^{t_{n+1}} \big[W(t_{n+1}) - W(s)\big]\,\mathrm{d}s = kW(t_{n+1}) - \boxed{\int_{t_n}^{t_{n+1}} W(s)\,\mathrm{d}s}.$$

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We then approximate the last term by $k^2 \sum_{\ell=1}^{k^{-1}} W(t_{n,\ell})$ to get a computable approximation of $\widetilde{\Delta_n W}$ by

$$\boxed{\widehat{\Delta_n W} := kW(t_{n+1}) - k^2 \sum_{\ell=1}^{k^{-1}} W(t_{n,\ell})}, \tag{20}$$

where $\{W(t_{n,\ell})\}_{\ell=1}^{k^{-1}}$ is the piecewise affine approximation of W on $[t_n, t_{n+1}]$ on an equidistant mesh $\{t_{n,\ell}\}_{\ell=1}^{k^{-1}}$, of step size $k^2 := t_{n,\ell+1} - t_{n,\ell}$.

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We have

$$\mathbb{E}\left[|\widetilde{\Delta_n W}|^2\right] \leq k \int_{t_n}^{t_{n+1}} \mathbb{E}\Big[|W(t_{n+1}) - W(s)|^2\Big] \,\mathrm{d}s \leq Ck^3 \,,$$

and the identity (20) infers for $q = 1, 2$

$$\begin{aligned} \mathbb{E}\left[|\widehat{\Delta_n W}|^{2q}\right] &\leq Ck^{2q}\, \mathbb{E}\big[|W(t_{n+1})|^{2q}\big] + Ck^{2q+1} \sum_{\ell=1}^{k^{-1}} \mathbb{E}\big[|W(t_{n,\ell})|^{2q}\big] \\ &\leq Ck^{3q} + Ck^{4q} \leq Ck^{3q} \,. \end{aligned}$$

Hence, the approximation of $\widetilde{\Delta_n W}$ by $\widehat{\Delta_n W}$ maintains the mean property of the former. This is the very reason to use k^2 as the step size to approximate the term.

Assumptions: Choice of u^1 and v^1

We choose $(u^0, v^0) = (u(0), v(0))$, together with

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We also assume:

∂D of class C^4, and $(u_0, v_0) \in (\mathbb{H}^1_0 \cap \mathbb{H}^4) \times (\mathbb{H}^1_0 \cap \mathbb{H}^3)\,.$

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The Tools for the Error Analysis

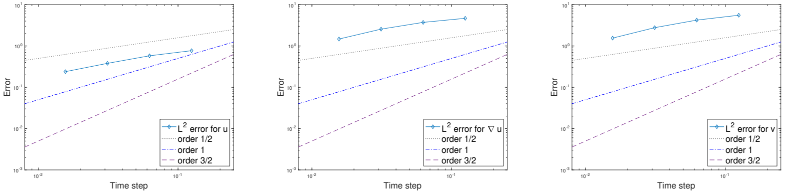
The core of the analysis is

- Hölder continuity in time of the solutions (for error analysis),
- Higher moment bounds for the solutions of the continuous problem,
- Higher moment bounds for the solutions of the discrete problem,
- Trapezoidal quadrature rule developed by Dragomir [5] (2000).

Some Interesting Computational Observations

Example 4

Consider $\sigma(v) = v$ and $F \equiv 0$. Fig. 3 displays convergence studies for the $(\widehat{\alpha}, \beta)$−scheme for $\widehat{\alpha} = 1$ and $\beta = 1/4$: the plots $(a) - (C)$ of $\mathbb{L}^2$-errors in $u, \nabla u$ and v, respectively, confirm convergence order $\mathcal{O}(k^{1/2})$.



(a) $\mathbb{L}^2$-error for u (b) $\mathbb{L}^2$-error for ∇u (c) $\mathbb{L}^2$-error for v

Figure 3: **(Example 4)** Rates of convergence of the $\left(1, \frac{1}{4}\right)$−scheme with $\sigma(v) = v$ and $F \equiv 0$.

Example 5

Consider the following case: $\sigma(u) = u$ and $F(u, v) = \cos(u) + 2v$;

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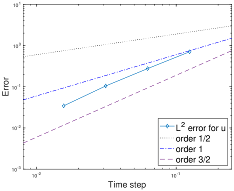
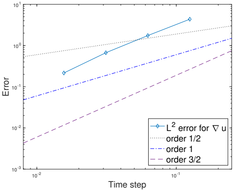
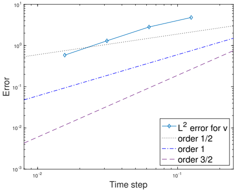


(d) $\mathbb{L}^2$-error for u



(e) $\mathbb{L}^2$-error for ∇u



(f) $\mathbb{L}^2$-error for v

Example 6

Consider the following case: $\sigma(u, v) = \frac{u}{1+u^2} + v$ and $F(u, v) = \cos(u) + 2v$;

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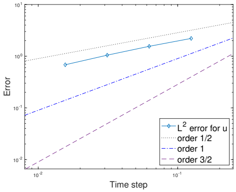
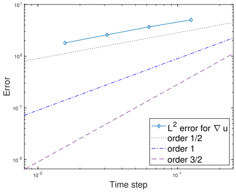
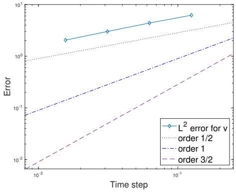


(j) $\mathbb{L}^2$-error for u



(k) $\mathbb{L}^2$-error for ∇u



(l) $\mathbb{L}^2$-error for v

Example 7

Let $F \equiv 0$, and drop the assumption on $\sigma \equiv \sigma(u)$ to be Lipschitz, *i.e.*, let $\sigma(u) = \sqrt{|u|}$.

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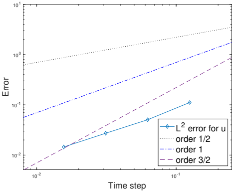
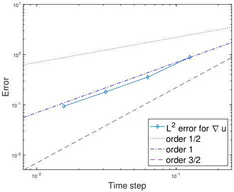
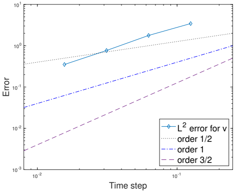


(p) $\mathbb{L}^2$-error for u



(q) $\mathbb{L}^2$-error for ∇u



(r) $\mathbb{L}^2$-error for v

Example 8

Let $D = (0,1)$, $T = 0.5$, $F \equiv 0$, $\sigma(v) = 5v$. For increased value of β, stabilization effect vanishes for small Δt. Thus, a smaller choice of β is preferred to have the stability of the scheme.

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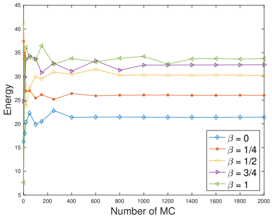
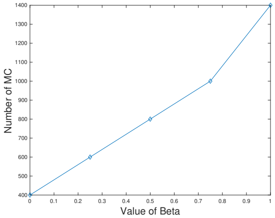


(a) Energy for different β



(b) β vs number of MC

Figure 4: The fig. (a) shows for $\beta = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$, that at least $\mathrm{MC} = 400, 600, 800, 1000, 1400$, are needed to have a steady of the energy $\mathcal{E}$ at time $T = 0.5$. The fig. (b) evidence a higher number of MC as we increase β to have a steady energy curve.

References

- R. Anton, D. Cohen, S. Larsson, and X. Wang, *Full Discretization of Semilinear Stochastic Wave Equations Driven by Multiplicative Noise*, SIAM J. Numer. Anal. **54**, No. 2: 1093–1119 (2016).
- P. Chow, Stochastic Partial Differential Equations, *Second edition, Chapman & Hall/CRC*, New York (2015).
- D. Cohen, and L. Quer-Sardanyons, *A fully discrete approximation of the one-dimensional stochastic wave equation*, IMA J. Numer. Anal., **36**: 400–420 (2016).
- R. Dalang, D. Khoshnevisan, C. Mueller, D. Nualart, and Y. Xiao, *A Minicourse on Stochastic Partial Differential Equations*, Lecture Notes in Math. 1962, Springer-Verlag, Berlin (2009).
- S. S. Dragomir and S. Mabizela, *Some error estimates in the trapezoidal quadrature rule*. Tamsui Oxf. J. Math. Sci., **16**(2): 259–272 (2000).
- T. Dupont, *L^2-estimates for Galerkin Methods for Second Order Hyperbolic Equations*. SIAM J. Numer. Anal., **10**(5): 880–889 (1973).
- L. Quer-Sardanyons, and M. Sanz-Solé, Space semi-discretisations for a stochastic wave equation, Potential Anal., **24**, 303—332 (2006).
- J. B. Walsh, *On numerical solutions of the stochastic wave equation*, Illinois J. Math., **50**: 991–1018 (2006).

Thank you for your kind attention!